THE STIELTJES TRANSFORM*

BY

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Introduction. The Stieltjes transform is defined by the equation

$$(1) \qquad f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} = \lim_{R\to\infty} \int_0^R \frac{d\alpha(t)}{x+t}.$$

We assume that $\alpha(t)$ is of bounded variation in $(0, R)$ for every positive R and that the limit (1) exists. If $\alpha(t)$ is the integral of a function $\phi(t)$, we obtain the special case

$$(2) \qquad f(x) = \int_0^\infty \frac{\phi(t)}{x+t}\, dt.$$

In this paper we discuss two distinct but related questions, the inversion problem and the solubility problem. In the former we assume that $f(x)$ is a function which admits the representation (1) or (2), and seek to determine $\alpha(t)$ or $\phi(t)$ from $f(x)$. In the latter we seek necessary and sufficient conditions on $f(x)$ that it should have the representation (1) or (2).

A solution of the inversion problem was given by Stieltjes† himself by means of contour integration. His result was

$$\frac{\alpha(t+) + \alpha(t-)}{2} = \lim_{\gamma\to 0} R\, \frac{1}{\pi i} \int_{-t-i\gamma}^{-i\gamma} f(s)\,ds,$$

where the symbol R means "real part of."

It is our purpose to obtain a real inversion formula, one depending only on a knowledge of $f(x)$ and its derivatives on the positive real axis. One may conjecture the existence of such a formula by noting that the Stieltjes transform is the product of two Laplace transforms. That is,

$$f(x) = \int_0^\infty e^{-xu} du \int_0^\infty e^{-ut} d\alpha(t).$$

But a Laplace transform admits of two types of inversion, one by contour integration and one by use of the successive derivatives of $f(x)$ on the positive real axis.‡ As we showed in the paper cited, these two inversions are

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† *Collected works of T. J. Stieltjes*, vol. 2, p. 473.

‡ D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, these Transactions, vol. 36 (1934), p. 107.

analogous to the two classical determinations of the coefficients of a power series, one by Cauchy's integral, the other by Taylor's series.

A real inversion of the integral (2) has been found recently by Paley and Wiener* in case $\phi(t)$ and $[\phi(t)]^2$ are integrable in the interval $(0, \infty)$. The result is

$$\text{(3)} \qquad \underset{m\to\infty}{\text{l.i.m.}}\ \frac{1}{\pi t^{1/2}} \sum_{n=0}^{m} \frac{(-1)^n}{(2n)!}\left(\pi t \frac{d}{dt}\right)^{2n}\left[t^{1/2} f(t)\right].$$

The formula obtained in the present paper seems, at first sight, totally unrelated, but we shall show later that this is not the case. It is

$$L_{k,t}[f(x)] = \frac{(-t)^{k-1}}{k!(k-2)!}\frac{d^{2k-1}}{dt^{2k-1}}\left[t^k f(t)\right].$$

This is a linear differential operator of order $2k-1$. With no restrictions on $\phi(t)$ beyond those necessary to make (2) converge we show that

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \phi(t)$$

for almost all positive t. Further, we prove that

$$\lim_{k\to\infty}\int_0^t L_{k,u}[f(x)]du = \frac{\alpha(t+) + \alpha(t-)}{2} - \alpha(0+)$$

for all positive t.

The method employed is the same as that used earlier by the author for the Laplace transform. That is, one employs a known method for discussing the asymptotic behavior of an integral of the form

$$\int_0^\infty [g(t)]^k \phi(t)dt$$

as k becomes infinite. In the present case

$$g(t) = \frac{t}{(x+t)^2},$$

a function which has a single maximum at $t=x$. We observe that the fundamental solutions of the linear differential expression $L_{k,t}[f(x)]$ are

$$f(x) = x^n \qquad (n = -k, -k+1, -k+2, \cdots, -1, 0, 1, \cdots, k-2).$$

Hence we may say that the Stieltjes transform is inverted by the linear differ-

* R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934.

ential operator of infinite order which has all integral powers of x as its fundamental solutions.

We solve the solubility problem by obtaining necessary and sufficient conditions that $f(x)$ should have the representation (1) or (2) with $\alpha(t)$ or $\phi(t)$ belonging to certain familiar classes. The most important classes considered are $\alpha(t)$ of bounded variation or non-decreasing, $\phi(t)$ of class $L^p;(p\geqq 1)$ or bounded. For example, we prove that $f(x)$ has the form (1) with $\alpha(t)$ non-decreasing if and only if

$$f(x) \geqq 0, \qquad (-1)^k[x^k f(x)]^{(2k-1)} \geqq 0 \qquad (x > 0;\ k = 1, 2, \cdots),$$
$$f(\infty) = 0.$$

This is the analogue of Bernstein's theorem on completely monotonic functions. The case of the integral (2) with $\phi(t)$ belonging to L^2 was treated by Paley and Wiener in the Colloquium lectures cited above.

A formula for computing the saltus of $\alpha(t)$ at a point of discontinuity is also obtained. In fact we show that

$$\lim_{k\to\infty} 2t\left(\frac{\pi}{k}\right)^{1/2} L_{k,t}[f(x)] = \alpha(t+) - \alpha(t-) \qquad (t > 0).$$

An inversion formula for the generalized Stieltjes transform

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{(x+t)^\rho} \qquad (\rho > 0)$$

is also obtained.

The final section of the paper shows the relation between the Paley-Wiener operator and $L_{k,t}[f(x)]$. The former can be written symbolically as

$$(\cos \pi \mathcal{D})[t^{1/2}f(t)],$$

where

$$\mathcal{D} = t\frac{d}{dt}\cdot$$

For, the finite series (3) is clearly a section of the infinite power series for $\cos \pi\mathcal{D}$. We show that $L_{k,t}[f(x)]$ is essentially a section of the familiar infinite product expansion of the cosine, so that symbolically the operators are equivalent. It should be observed that the Paley-Wiener operator is applicable only to the case in which $\phi(t)$ belongs to L^2 (at least in so far as proofs have yet been given), whereas the operator of the present paper is not so restricted.

1. **General properties.** Let $\alpha(t)$ be a complex function of the real variable t of bounded variation in the interval $0 \leqq t \leqq R$ for every positive R. Such a function is said to be normalized if

$$\alpha(0) = 0,$$
$$\alpha(t) = \frac{\alpha(t+) + \alpha(t-)}{2} \qquad (t > 0).$$

We assume throughout that $\alpha(t)$ has these properties. It is clear that the integral

$$\int_0^R \frac{d\alpha(t)}{s+t}$$

exists for every complex $s = \sigma + i\tau$ not on the negative real axis, $\tau = 0$, $\sigma \leqq 0$, which we shall henceforth denote by D. Set

$$f(s) = \int_0^\infty \frac{d\alpha(t)}{s+t} = \lim_{R\to\infty} \int_0^R \frac{d\alpha(t)}{s+t} \tag{1.1}$$

whenever the indicated limit exists. Then the improper integral (1.1) is said to converge. We show at once that the region of convergence of (1.1) is the whole complex plane less the ray D.

THEOREM 1.1. *If the integral* (1.1) *converges for a point* s_0 *not on* D, *it converges for every such point.*

For, set

$$\beta(0) = 0, \quad \beta(t) = \int_0^t \frac{d\alpha(u)}{s_0 + u} \qquad (t > 0).$$

Then for s not on D

$$\int_0^R \frac{d\alpha(t)}{s+t} = \int_0^R \frac{s_0+t}{s+t}\, d\beta(t) = \beta(R)\frac{s_0+R}{s+R} + (s_0 - s)\int_0^R \frac{\beta(t)}{(s+t)^2}\, dt.$$

Since $\beta(R)$ approaches a limit as R becomes infinite, it is clear that the integral

$$\int_0^\infty \frac{\beta(t)}{(s+t)^2}\, dt$$

converges absolutely and that (1.1) converges. Moreover,

$$\int_0^\infty \frac{d\alpha(t)}{s+t} = \int_0^\infty \frac{d\alpha(t)}{s_0+t} + (s_0 - s)\int_0^\infty \frac{\beta(t)}{(s+t)^2}\, dt. \tag{1.2}$$

We observe that (1.1) may converge without converging absolutely as the example.

$$f(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{s+n}$$

shows. But (1.2) always enables us to replace the given integral by an absolutely convergent one.

At this point we note a contrast with the theory of the Laplace transform. In the latter case a direct integration by parts replaces a conditionally convergent integral by an absolutely convergent one. That this is not always true for the Stieltjes transform is seen by an example. Take

$$\alpha(0) = 0,$$
$$\alpha(t) = (-1)^n \frac{t+2}{\log(t+2)} \qquad (n < t < n+1;\ n = 0, 1, 2, \cdots).$$

Then

$$\int_0^{\infty} \frac{|\alpha(t)|}{(t+2)^2}\, dt = \int_0^{\infty} \frac{dt}{(t+2)\log(t+2)}$$

clearly diverges. But

$$\int_0^{\infty} \frac{d\alpha(t)}{t+2} = \int_0^{\infty} \frac{\alpha(t)}{(t+2)^2}\, dt = \sum_{n=0}^{\infty} (-1)^n \log \frac{\log(n+3)}{\log(n+2)},$$

the series converging. For this example, integration by parts replaces a conditionally convergent integral by another with the same property.

COROLLARY 1.11. *If* (1.1) *converges, it converges uniformly in any bounded closed region not containing a point of D.*

COROLLARY 1.12. *If* (1.1) *converges, $f(s)$ is analytic at points not on D.*

COROLLARY 1.13. *If* (1.1) *converges,*

$$f^{(k)}(s) = (-1)^k k! \int_0^{\infty} \frac{d\alpha(t)}{(s+t)^{k+1}} \qquad (k = 0, 1, \cdots).$$

Another very useful result is contained in

THEOREM 1.2. *If* (1.1) *converges, then*

$$\alpha(t) = o(t) \qquad (t \to \infty). \tag{1.3}$$

Let (1.1) converge for $s = s_0$ not on D, and define $\beta(t)$ as in Theorem 1.1. Then

$$\alpha(R) = \int_0^R d\alpha(t) = \int_0^R (t + s_0)d\beta(t).$$

But

$$\int_0^R td\beta(t) = R\beta(R) - \int_0^R \beta(t)dt,$$

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R td\beta(t) = \beta(\infty) - \beta(\infty) = 0.$$

Also

$$\lim_{R\to\infty} \frac{1}{R}\int_0^R s_0 d\beta(t) = 0,$$

so that (1.3) is established.

COROLLARY 1.21. *If* (1.1) *converges, then*

$$f(s) = \int_0^\infty \frac{\alpha(t)}{(s+t)^2}\, dt.$$

It is important to note that the converse of Theorem 1.2 is false. Thus (1.3) holds if

$$\alpha(t) = \int_0^t \frac{du}{\log (u+2)} \qquad (t \geqq 0).$$

Yet for this definition of $\alpha(t)$ the integral (1.1) diverges.

However, it is easily seen that if

$$\alpha(t) = O(t^{1-\delta}) \qquad (\delta > 0,\ t \to \infty),$$

then (1.1) converges.

The relation between the Laplace and Stieltjes transforms is made precise in

THEOREM 1.3. *If the integral* (1.1) *converges, then*

$$(1.4) \qquad f(s) = \int_{0+}^\infty e^{-st}\phi(t)dt \qquad (\sigma > 0),$$

*where**

$$(1.5) \qquad \phi(t) = \int_0^\infty e^{-tu}d\alpha(u) \qquad (t > 0).$$

* The notation employed in (1.4) means that $\int_{0+}^\infty g(t)dt = \lim_{\epsilon\to 0} \int_\epsilon^\infty g(t)dt$.

For, by Theorem 1.2 $\alpha(u)=o(u)$ when u becomes infinite, so that (1.5) converges uniformly* in the closed interval $\epsilon \leqq t \leqq R$ for arbitrary positive numbers ϵ and R. Hence

$$\int_\epsilon^R e^{-st}dt \int_0^\infty e^{-tu}d\alpha(u) = \int_0^\infty \frac{e^{-(s+u)\epsilon} - e^{-(s+u)R}}{s+u}\, d\alpha(u).$$

If s is any point not on the ray D, the integral

$$\int_0^\infty \frac{e^{-\epsilon u}}{s+u}\, d\alpha(u)$$

clearly converges, so that

$$(1.6)\quad \int_\epsilon^R e^{-st}dt \int_0^\infty e^{-tu}d\alpha(u) = e^{-s\epsilon}\int_0^\infty \frac{e^{-\epsilon u}}{s+u}\, d\alpha(u) - e^{-sR}\int_0^\infty \frac{e^{-uR}}{s+u}\, d\alpha(u).$$

The first integral on the right-hand side converges uniformly† (s being fixed) in the interval $0 \leqq \epsilon < \infty$, and hence approaches $f(s)$ as ϵ approaches zero. Moreover,†

$$\lim_{R\to\infty}\int_0^\infty \frac{e^{-uR}}{s+u}\, d\alpha(u) = \lim_{t\to 0+}\int_0^t \frac{d\alpha(u)}{s+u} = \frac{\alpha(0+)}{s},$$

so that the last term of (1.6) approaches zero with $1/R$ if $\sigma>0$. Our result is consequently established by allowing ϵ to approach zero and R to become infinite.

Note that the inequality $\sigma>0$ in (1.4) cannot be replaced by $\sigma \geqq 0$ as the example $f(s)=1/s$ shows. We observe also that the converse of Theorem 1.3 is not true. That is, the integrals (1.4) and (1.5) may converge in the range specified without having $f(s)$ represented in the form (1.1). For example, take

$$\phi(t) = \frac{1}{(1+e^{-t})^2} = \sum_{n=0}^\infty (-1)^n(n+1)e^{-nt} \qquad (t>0),$$

$$f(s) = \int_0^\infty \frac{e^{-st}}{(1+e^{-t})^2}\, dt \qquad (\sigma>0).$$

The integral (1.1) becomes

$$\sum_{n=0}^\infty (-1)^n \frac{(n+1)}{s+n},$$

* See footnote on p. 12.

† For the results regarding the Laplace transform which are here employed see, for example, D. V. Widder, *A generalization of Dirichlet's series and of Laplace's integrals by means of a Stieltjes integral*, these Transactions, vol. 31 (1929), p. 694.

a series which diverges for all s. In this connection we may prove

THEOREM 1.4. *If* $\alpha(u)$ *is such that the integral* (1.5) *converges for* $t \geqq 0$, *then the function* $f(s)$ *defined by* (1.4) *also has the representation* (1.1) *for all* s *not on* D.

For, in this case, $\alpha(u)$ is necessarily bounded and we may apply Theorem 1.3.

A similar result is contained in

THEOREM 1.5. *If* $\alpha(u)$ *is non-decreasing and such that* (1.4) *and* (1.5) *converge for* $\sigma > 0$, $t > 0$ *respectively, then the function* $f(s)$ *defined by* (1.4) *has the representation* (1.1) *for all* s *not on* D.

The proof is easily supplied. We turn next to the uniqueness theorem.

THEOREM 1.6. *If the normalized function* $\alpha(t)$ *is such that*

$$\int_0^\infty \frac{d\alpha(t)}{s_0 + t + nl} = 0 \qquad (l > 0,\ n = 0, 1, 2, \cdots),$$

then

$$\alpha(t) = 0 \qquad (0 \leqq t < \infty).$$

This follows from Theorem 1.3 and from Lerch's uniqueness theorem for Laplace integrals.*

We conclude this section with a proof of

THEOREM 1.7. *If* $\alpha(t)$ *is a real non-decreasing function for which the point* $t = 0$ *is a point of increase and for which the integral*

$$f(s) = \int_0^\infty \frac{d\alpha(t)}{s + t}$$

converges, then $f(s)$ *has a singularity at* $s = 0$.

For, suppose the contrary. Then the series

$$f(s) = \sum_{n=0}^{\infty} f^{(n)}(1) \frac{(s-1)^n}{n!}$$

converges for some point on the negative real axis, say $s = -\epsilon$, and

$$f(-\epsilon) = \sum_{n=0}^{\infty} (-1)^n f^{(n)}(1) \frac{(\epsilon + 1)^n}{n!}.$$

Applying Corollary 1.13 we obtain

* The usual proof of this theorem can easily be extended to include Cauchy-values of Laplace integrals: $\int_{0+}^{\infty} e^{-xt} d\alpha(t)$.

$$f(-\epsilon) = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{(\epsilon+1)^n}{(t+1)^{n+1}} d\alpha(t)$$
$$= \sum_{n=0}^{\infty} (n+1) \int_0^{\infty} \frac{(\epsilon+1)^n}{(t+1)^{n+2}} \alpha(t)dt.$$

This series dominates the series

$$\sum_{n=0}^{\infty} (n+1) \int_{\epsilon}^{\infty} \frac{(\epsilon+1)^n}{(t+1)^{n+2}} \alpha(t)dt, \tag{1.7}$$

so that the latter also converges. Since the integrand is non-negative, and since the series

$$\sum_{n=0}^{\infty} (n+1) \frac{(\epsilon+1)^n}{(t+1)^{n+2}} = \frac{1}{(t-\epsilon)^2}$$

converges for $t>\epsilon$, we may interchange integral and summation symbols in (1.7). That is, the integral

$$\int_{\epsilon}^{\infty} \frac{\alpha(t)}{(t-\epsilon)^2} dt \tag{1.8}$$

must converge. But since $t=0$ is a point of increase for $\alpha(t)$, it follows that $\alpha(\epsilon+)>0$ and

$$\lim_{t\to\epsilon+} \frac{\alpha(t)}{t-\epsilon} = +\infty.$$

Hence (1.8) can not converge. The assumption that $f(s)$ is analytic at $s=0$ is untenable. That is, $s=0$ is a singularity of $f(s)$.

2. **Inversion in a special case.** The results of the previous section enable us to restrict attention to the real variable $s=x$. In fact we shall even assume that $\alpha(t)$ is a real function. The loss of generality thus involved is trivial. The reader who has need of results for complex functions $\alpha(t)$ has only to apply the theorems proved to the real and imaginary parts of $\alpha(t)$ separately.

We introduce a functional operator by

DEFINITION 2.1. *An operator $L_{k,t}[f(x)]$ is defined by the equations*

$$L_{0,t}[f(x)] = c_0 f(t),$$
$$L_{k,t}[f(x)] = c_k(-t)^{k-1}[t^k f(t)]^{(2k-1)} \quad (k=1,2,3,\cdots),$$

where

$$c_0 = c_1 = 1, \quad c_k = \frac{1}{k!(k-2)!} \quad (k=2,3,4,\cdots).$$

Obviously the operator can be applied only to functions which possess derivatives of order $(2k-1)$. It becomes of interest only when k is allowed to become infinite, so that we shall be applying it only to functions which possess derivatives of all orders. Our first application will be to the function (1.1) where $\alpha(t)$ is a step-function defined as follows:

$$\alpha(t) = \begin{cases} 0 & (0 \leqq t < a), \\ 1 & (a < t < \infty), \end{cases}$$

$$\alpha(a) = \tfrac{1}{2}.$$

In this case

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} = \frac{1}{x+a} \cdot$$

Simple computation gives

$$L_{k,t}[f(x)] = d_k \frac{t^{k-1}a^k}{(t+a)^{2k}} \qquad (k = 1, 2, 3, \cdots),$$

$$d_k = (2k-1)!c_k.$$

We now prove

THEOREM 2.1. *If $a>0$, $t>0$, then*

$$\lim_{k\to\infty} \int_0^t L_{k,u}\left[\frac{1}{x+a}\right] du = \begin{cases} 0 & (0 \leqq t < a), \\ \frac{1}{2} & (t = a), \\ 1 & (1 < t < \infty). \end{cases}$$

That is, the operator $L_{k,t}[f(x)]$ serves to invert the integral (1.1) at least in this special case. Set

$$H_k(t) = d_k \int_0^t \frac{u^{k-1}}{(u+1)^{2k}} du.$$

Then

$$\int_0^t L_{k,u}\left[\frac{1}{x+a}\right] du = H_k\left(\frac{t}{a}\right),$$

so that we need prove only

$$\lim_{k\to\infty} H_k(t) = \begin{cases} 0 & (0 \leqq t < 1), \\ \frac{1}{2} & (t = 1), \\ 1 & (1 < t < \infty). \end{cases}$$

If $0 \leqq t < 1$ we have

$$0 \leqq H_k(t) < d_k \left[\frac{t}{(t+1)^2}\right]^{k-1}. \tag{2.1}$$

Since

$$\lim_{k\to\infty} \frac{(2k-1)(2k-2)}{k(k-2)} \frac{t}{(t+1)^2} = \frac{4t}{(t+1)^2} < 1,$$

it follows that the extreme right-hand member of (2.1), and hence also $H_k(t)$, tends to zero with $1/k$.

If $1 < t < \infty$, we have by use of the B-function

$$\frac{k-1}{k} - H_k(t) = d_k \int_t^\infty \frac{u^{k-1}}{(u+1)^{2k}} du.$$

But

$$0 < d_k \int_t^\infty \frac{u^{k-1}}{(u+1)^{2k}} du < d_k \left[\frac{t}{(t+2)^2}\right]^{k-1},$$

so that in this case

$$\lim_{k\to\infty} H_k(t) = 1.$$

Finally, if $t=1$,

$$H_k(1) = \frac{k-1}{k} - d_k \int_1^\infty \frac{u^{k-1}}{(u+1)^{2k}} du = \frac{k-1}{k} - d_k \int_0^1 \frac{u^{k-1}}{(u+1)^{2k}} du$$
$$= \frac{k-1}{k} - H_k(1)$$

by an obvious change of variable. Hence

$$H_k(1) = \frac{1}{2}\,\frac{k-1}{k} \to \frac{1}{2} \qquad (k \to \infty).$$

This completes the proof of the theorem.

3. **The inversion of the general Stieltjes integral.** Before proceeding to the general case we need to prove the following simple, but extremely useful lemma:

LEMMA 3.11. *If $f(x)$ has a derivative of order $(2k-1)$, then*

$$x^{k-1}\left[x^k f(x)\right]^{(2k-1)} = \left[x^{2k-1} f^{(k-1)}(x)\right]^{(k)}.$$

The proof consists merely in computing both sides of the equation by Leibniz's rule. In each case we obtain

$$\sum_{p=0}^{k} \frac{(2k-1)!k!}{(2k-p-1)!p!(k-p)!} f^{(2k-p-1)}(x) x^{2k-p-1}.$$

We shall also need

LEMMA 3.12. *If* (1.1) *converges, then*

$$\lim_{x\to 0+} d_k x^k \int_0^\infty \frac{t^{k-1}\alpha(t)}{(x+t)^{2k}}\,dt = \alpha(0+)\,\frac{k-1}{k} \qquad (k = 2, 3, \cdots).$$

For, if ϵ is a given positive number, we determine $\delta(\epsilon)$ such that

$$|\alpha(t) - \alpha(0+)| < \epsilon \qquad 0 \leqq t \leqq \delta(\epsilon).$$

Then

$$\left| d_k x^k \int_0^\infty \frac{t^{k-1}[\alpha(t) - \alpha(0+)]}{(x+t)^{2k}}\,dt \right| \leqq \epsilon d_k x^k \int_0^{\delta(\epsilon)} \frac{t^{k-1}}{(x+t)^{2k}}\,dt + \left| d_k x^k \int_{\delta(\epsilon)}^\infty \frac{t^{k-1}}{(x+t)^{2k}}[\alpha(t) - \alpha(0+)]dt \right|, \tag{3.1}$$

$$\limsup_{x\to 0+} \left| d_k x^k \int_0^\infty \frac{t^{k-1}[\alpha(t) - \alpha(0+)]}{(x+t)^{2k}}\,dt \right| \leqq \epsilon\,\frac{k-1}{k} < \epsilon,$$

the second term on the right-hand side of (3.1) clearly approaching zero with x. Hence

$$\lim_{x\to 0+} d_k x^k \int_0^\infty \frac{t^{k-1}}{(x+t)^{2k}}\,\alpha(t)dt = d_k\alpha(0+)x^k \int_0^\infty \frac{t^{k-1}}{(x+t)^{2k}}\,dt = \alpha(0+)\,\frac{k-1}{k}.$$

In a similar way we can prove

LEMMA 3.13. *If* $\alpha(\infty)$ *exists, then*

$$\lim_{x\to\infty} d_k x^k \int_0^\infty \frac{t^{k-1}}{(x+t)^{2k}}\,\alpha(t)dt = \alpha(\infty)\,\frac{k-1}{k} \qquad (k = 2, 3, 4, \cdots).$$

By use of these results we can now establish

THEOREM 3.1. *If the integral* (1.1) *converges, then*

$$\lim_{k\to\infty} \int_0^t L_{k,u}[f(x)]du = \alpha(t) - \alpha(0+) \qquad (t > 0).$$

We begin by writing the integral (1.1) in the form

$$f(x) = \int_0^\infty \frac{\alpha(t)}{(x+t)^2}\,dt,$$

which we are enabled to do by Corollary 1.21. By Lemma 3.11 we have

$$\int L_{k,u}[f(x)]du = c_k[u^{2k-1}f^{(k-1)}(u)]^{(k-1)}.$$

But simple computation gives

$$(-1)^{k-1}c_k\left[u^{2k-1}f^{(k-1)}(u)\right]^{(k-1)} = d_k u^k \int_0^\infty \frac{y^{k-1}\alpha(y)}{(u+y)^{2k}}dy.$$

Hence by Lemma 3.12

$$\int_0^t L_{k,u}[f(x)]du = d_k t^k \int_0^\infty \frac{y^{k-1}\alpha(y)}{(t+y)^{2k}}dy - \frac{k-1}{k}\alpha(0+).$$

Consequently, it remains only to show that

$$\lim_{k\to\infty} d_k t^k \int_0^\infty \frac{y^{k-1}\alpha(y)}{(t+y)^{2k}}dy = \alpha(t) \qquad (t>0). \tag{3.2}$$

Set $y/t=v$. The integral in question becomes

$$d_k \int_0^\infty \frac{v^{k-1}\alpha(tv)}{(1+v)^{2k}}dv.$$

Further, set

$$\psi_t(v) = \begin{cases} \alpha(t-) & (0<v<1), \\ \alpha(t) & (v=1), \\ \alpha(t+) & (1<v<\infty). \end{cases}$$

Then*

$$d_k \int_0^\infty \frac{v^{k-1}}{(1+v)^{2k}}\psi_t(v)dv = [\alpha(t+)+\alpha(t-)]H_k(1).$$

Hence by Theorem 2.1

$$\lim_{k\to\infty} d_k \int_0^\infty \frac{v^{k-1}}{(1+v)^{2k}}\psi_t(v)dv = \frac{\alpha(t+)+\alpha(t-)}{2} = \alpha(t).$$

Set

$$\beta_t(v) = \alpha(tv) - \psi_t(v).$$

This function is continuous at $t=1$ and vanishes there. It remains only to show that

$$\lim_{k\to\infty} d_k \int_0^\infty \frac{v^{k-1}}{(1+v)^{2k}}\beta_t(v)dv = 0.$$

Given $\epsilon>0$, we determine $\delta(\epsilon)$ such that

* For the definition of $H_k(t)$ see §2.

$$|\beta_t(v)| < \epsilon \qquad |v-1| \leqq \delta(\epsilon).$$

Then

$$\left| d_k \int_0^\infty \frac{v^{k-1}}{(1+v)^{2k}} \beta_k(v)dv \right| \leqq O(1)H_k(1-\delta) + \epsilon\{H_k(1+\delta) - H_k(1-\delta)\} \tag{3.3}$$
$$+ O(1)\left\{\frac{k-2}{k-1} - H_{k-1}(1+\delta)\right\}.$$

In obtaining the last term on the right-hand side of (3.3), we have used the obvious fact that

$$\beta_t(v) = O(v) \qquad (v \to \infty).$$

Letting k become infinite in (3.3) and making use of Theorem 2.1, we have

$$\limsup_{k\to\infty} \left| d_k \int_0^\infty \frac{v^{k-1}}{(1+v)^{2k}} \beta_t(v)dv \right| \leqq \epsilon,$$

from which our result follows at once.

COROLLARY 3.11. *If* (1.1) *converges and if* $\alpha(t)$ *is continuous in* (a, b), *then*

$$\lim_{k\to\infty} \int_0^t L_{k,u}[f(x)]du = \alpha(t) - \alpha(0+)$$

uniformly in the interval $\alpha \leqq t \leqq \beta$, *where*

$$a < \alpha < \beta < b \qquad (a > 0),$$
$$a \leqq \alpha < \beta < b \qquad (a = 0).$$

To prove this one has only to show that

$$\lim_{k\to\infty} d_k t^k \int_0^\infty \frac{y^{k-1}}{(t+y)^{2k}} [\alpha(y) - \alpha(t)]dy$$
$$= \lim_{k\to\infty} d_k \int_0^\infty \frac{y^{k-1}}{(t+y)^{2k}} [\alpha(ty) - \alpha(t)]dy = 0$$

uniformly in (α, β). If we note that

$$\alpha(ty) - \alpha(t) = o(1) \qquad (y \to 1)$$

uniformly in t for t in (α, β), the proof of this proceeds as for Theorem 3.1.

COROLLARY 3.12. *If* $\alpha(\infty)$ *exists, then*

$$\lim_{k\to\infty} \int_0^\infty L_{k,u}[f(x)]du = \alpha(\infty) - \alpha(0+).$$

COROLLARY 3.13. *If*

$$\alpha(t) = O(t^\rho) \qquad (t \to \infty)$$

for some positive value of ρ, *then equation* (3.2) *holds.*

4. **The Lebesgue integral.** In this section we consider the special case of (1.1) in which

$$\alpha(t) = \int_0^t \phi(u)du,$$

the function $\phi(u)$ being integrable in $(0, R)$ for every positive R and of such a nature that (1.1) converges. We then have

$$f(x) = \int_0^\infty \frac{\phi(t)}{x+t}\, dt. \tag{4.1}$$

We now show that the operator $L_{k,t}[f(x)]$ serves to invert this integral also.

THEOREM 4.1. *If* t *is a point of the Lebesgue set for the function* $\phi(u)$, *and if* (4.1) *converges, then*

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \phi(t). \tag{4.2}$$

By the Lebesgue set we mean those points t for which

$$\int_0^h |\phi(u+t) - \phi(t)|\, du = o(h) \qquad (h \to 0).$$

Direct computation gives

$$L_{k,t}[f(x)] = d_k t^{k-1} \int_0^\infty \frac{u^k}{(t+u)^{2k}} \phi(u)du.$$

We have to show that

$$\lim_{k\to\infty} d_k t^{k-1} \int_0^\infty \frac{u^k}{(t+u)^{2k}} [\phi(u) - \phi(t)]du = \lim_{k\to\infty} d_k \int_0^\infty \frac{u^k}{(u+1)^{2k}} [\phi(tu) - \phi(t)]du = 0.$$

Set

$$\beta(y) = \int_1^y [\phi(ut) - \phi(t)]du.$$

Since t is a point of the Lebesgue set, we have

$$\beta(y) = o(|1-y|) \qquad (y \to 1). \tag{4.3}$$

Then the integral in question becomes

$$I_k = d_k \int_0^\infty \frac{u^k}{(u+1)^{2k}}\, d\beta(u) = d_k k \int_0^\infty \frac{(u-1)u^{k-1}}{(u+1)^{2k+1}}\, \beta(u) du.$$

If we note that

$$kd_k \int_{1-\delta}^{1+\delta} \frac{(u-1)^2 u^{k-1}}{(u+1)^{2k+1}}\, du < kd_k \int_0^\infty \frac{(u-1)^2 u^{k-1}}{(u+1)^{2k+1}}\, du = 1$$

and take account of (4.3), we have, by a method similar to that used in obtaining (3.2),

$$| I_k | \leqq O(1) H_k[1 - \delta(\epsilon)] + \epsilon + O(1)\left\{\frac{k-2}{k-1} - H_{k-1}[1 + \delta(\epsilon)]\right\},$$

$$\limsup_{k\to\infty} | I_k | \leqq \epsilon,$$

$$\lim_{k\to\infty} I_k = 0.$$

This proves the theorem.

COROLLARY 4.11. *Equation* (4.2) *holds for points t at which $\phi(t)$ is continuous.*

COROLLARY 4.12. *Equation* (4.2) *holds almost everywhere.*

COROLLARY 4.13. *If* (4.1) *converges and if $\phi(t)$ is continuous in (a, b), equation* (4.2) *holds uniformly in $\alpha \leqq t \leqq \beta$ where*

$$a < \alpha < \beta < b.$$

COROLLARY 4.14. *If $\phi(t+)$ and $\phi(t-)$ exist, then*

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \frac{\phi(t+) + \phi(t-)}{2}.$$

For, set

$$\beta(t, u) = \int_1^t [\phi(yu) - \phi(u)] dy,$$

and note that

$$\beta(t, u) = o(| 1 - t |) \qquad (t \to 1)$$

uniformly in $\alpha \leqq u \leqq \beta$. The proof is now completed by obvious modification of the proof of Theorem 4.1.

At this point we illustrate Theorem 4.1 by an example. Take

$$\phi(t) = \frac{t^{-\delta}}{\Gamma(1-\delta)\Gamma(\delta)} \qquad (0 < \delta < 1).$$

Then

$$f(x) = x^{-\delta}.$$

Simple computation gives

$$L_{k,t}[x^{-\delta}] = \frac{\Gamma(k-\delta+1)\Gamma(k+\delta-1)}{\Gamma(k+1)\Gamma(k-1)} \cdot \frac{t^{-\delta}}{\Gamma(1-\delta)\Gamma(\delta)} \cdot$$

But

$$\frac{\Gamma(k+\alpha)}{\Gamma(k)} \sim k^{\alpha} \qquad (\alpha > 0,\ k \to \infty),$$

so that we can prove directly that

$$\lim_{k\to\infty} L_{k,t}[x^{-\delta}] = \frac{t^{-\delta}}{\Gamma(1-\delta)\Gamma(\delta)} = \phi(t)$$

for all positive t.

5. **The saltus operator.** We now introduce a new operator by the

DEFINITION 5.1. *The operator $l_{k,t}[f(x)]$ is defined by the equation*

$$l_{k,t}[f(x)] = 2t\left(\frac{\pi}{k}\right)^{1/2} L_{k,t}[f(x)].$$

We first apply the operator to the spècial function

$$f(x) = \int_0^\infty \frac{d\psi_t(v)}{x+v} = \frac{\alpha(t+)-\alpha(t-)}{x+1},$$

where $\psi_t(v)$ was defined in §3. Direct computation gives

$$l_{k,u}[f(x)] = 2\left(\frac{\pi}{k}\right)^{1/2} d_k \frac{u^k}{(u+1)^{2k}} \left[\alpha(t+)-\alpha(t-)\right].$$

Then by use of Stirling's formula, or otherwise, we prove

LEMMA 5.11.

$$\lim_{k\to\infty} l_{k,u}\left[\frac{1}{x+1}\right] = \begin{cases} 0, & u \neq 1, \\ 1, & u = 1. \end{cases}$$

Hence, the limit of $l_{k,t}[f(x)]$ is the saltus of $\psi_t(u)$ at every point u. This result is general, as we now prove.

THEOREM 5.1. *If* (1.1) *converges, then*

$$\lim_{k\to\infty} l_{k,t}[f(x)] = \alpha(t+) - \alpha(t-) \qquad (t>0).$$

If we define $\beta(v)=\beta_t(v)$ as in §3 and note that

$$l_{k,t}[f(x)] = 2t^k\left(\frac{\pi}{k}\right)^{1/2} d_k \int_0^\infty \frac{u^k}{(u+t)^{2k}}\,d\alpha(u)$$
$$= 2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_0^\infty \frac{v^k}{(v+1)^{2k}}\,d\alpha(vt),$$

the special example treated above shows us that we need only prove

$$\lim_{k\to\infty} 2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_0^\infty \frac{v^k}{(v+1)^{2k}}\,d\beta(v) = 0,$$

the function $\beta(v)$ being continuous and equal to zero at $v=1$. If we integrate by parts, the integral in question becomes

$$I_k = -2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_0^\infty \beta(v)\frac{d}{dv}\frac{v^k}{(v+1)^{2k}}\,dv.$$

Now note that

$$2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_{1-\delta}^{1+\delta} \left|\frac{d}{dv}\frac{v^k}{(v+1)^{2k}}\right| dv < 2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_0^1 \frac{d}{dv}\frac{v^k}{(v+1)^{2k}}\,dv$$
$$-2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_1^\infty \frac{d}{dv}\frac{v^k}{(v+1)^{2k}}\,dv = 4\left(\frac{\pi}{k}\right)^{1/2}\frac{d_k}{2^{2k}} = 2l_{k,1}\left[\frac{1}{x+1}\right].$$

Hence, proceeding as in §3, we obtain

$$|I_k| \leqq O(1)l_{k,1-\delta}\left[\frac{1}{x+1}\right] + 2\epsilon l_{k,1}\left[\frac{1}{x+1}\right]$$
$$+ O(1)2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_{1+\delta}^\infty u\left|\frac{d}{du}\frac{u^k}{(u+1)^{2k}}\right| du.$$

Consequently

$$\limsup_{k\to\infty} |I_k| \leqq 2\epsilon,$$

$$\lim_{k\to\infty} I_k = 0,$$

since

$$2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_{1+\delta}^\infty u\left|\frac{d}{du}\frac{u^k}{(u+1)^{2k}}\right| du$$

$$= (1+\delta)l_{k,1+\delta}\left[\frac{1}{x+1}\right] + 2\left(\frac{\pi}{k}\right)^{1/2} d_k \int_{1+\delta}^{\infty} \frac{u^k}{(u+1)^{2k}}\,du = o(1) \quad (k \to \infty).$$

This completes the proof of the theorem.

The same type of argument enables us to prove the following related result.

THEOREM 5.2. *If* (1.1) *converges and if* $\alpha(t)$ *has right-hand and left-hand derivatives* $\alpha_+'(t)$ *and* $\alpha_-'(t)$ *respectively at a point* t, *then*

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \frac{\alpha_+'(t) + \alpha_-'(t)}{2}.$$

For, if we define

$$\omega(v) = \begin{cases} (v-1)\alpha_-'(t) & 0 \leqq v \leqq 1, \\ (v-1)\alpha_+'(t) & 1 \leqq v < \infty, \end{cases}$$

$$\gamma(v) = \alpha(vt) - \omega(v),$$

it is clear that

$$\gamma(v) = o(|1-v|) \quad (v \to 1), \tag{5.1}$$

and that*

$$\lim_{k\to\infty} d_k \int_0^{\infty} \frac{v^k}{(v+1)^{2k}}\,d\omega(v) = \frac{\alpha_+'(t) + \alpha_-'(t)}{2},$$

so that we have now to show that the integral

$$I_k = d_k \int_0^{\infty} \gamma(v) \frac{d}{dv} \frac{v^k}{(v+1)^{2k}}\,dv$$

approaches zero with $1/k$. This may be done by use of (5.1) if we note that

$$d_k \int_{1-\delta}^{1+\delta} |1-v| \left| \frac{d}{dv} \frac{v^k}{(v+1)^{2k}} \right| dv \leqq d_k \int_0^1 (1-v) \frac{d}{dv} \frac{v^k}{(v+1)^{2k}}\,dv$$

$$+ d_k \int_1^{\infty} (1-v) \frac{d}{dv} \frac{v^k}{(v+1)^{2k}}\,dv$$

$$= d_k \int_0^{\infty} (1-v) \frac{d}{dv} \frac{v^k}{(v+1)^{2k}}\,dv = d_k \int_0^{\infty} \frac{v^k}{(v+1)^{2k}}\,dv = 1.$$

COROLLARY 5.21. *If* $\alpha(t)$ *is constant in* (a, b), *then*

* This may be conveniently proved by breaking the interval into two parts corresponding to the intervals (0, 1) and $(1, \infty)$ and by using Corollary 4.14.

$$\lim_{k\to\infty} L_{k,t}[f(x)] = 0$$

uniformly for t in (α, β),

$$a < \alpha < \beta < b \qquad (a > 0),$$

$$a \leqq \alpha < \beta < b \qquad (a = 0).$$

Theorem 5.2 becomes of particular interest if $\alpha(t)$ is a function which is not an integral but which has a derivative almost everywhere. The integral (1.1) can not then be put in the form (4.1). Yet the inversion formula has the same effect as if $f(x)$ had the form (4.1) with $\alpha'(t)=\phi(t)$.

6. **Generalizations.** We turn now to a group of theorems which may be regarded as generalizations of our inversion formulas.

THEOREM 6.1. *If $\alpha(t)$ is of bounded variation in the interval* $(0, \infty)$, *and if $\psi(t)$ is an arbitrary function continuous* in the interval* $0 \leqq t \leqq \infty$, *then*

$$\lim_{k\to\infty}\int_0^\infty \psi(u)L_{k,u}[f(x)]du = \int_0^\infty \psi(u)d\alpha(u) - \alpha(0+)\psi(0). \tag{6.1}$$

Since we have already established that

$$\lim_{k\to\infty}\int_0^t L_{k,u}[f(x)]du = \alpha(t) - \alpha(0+),$$

we have only to take the limit under the integral sign on the left-hand side of (6.1) to obtain our result. This will be permissible by the Helly-Bray theorem† if the functions

$$\int_0^t L_{k,u}[f(x)]du$$

are of uniformly bounded variation in $(0, \infty)$ for $k=1, 2, 3, \cdots$. This is so under our hypotheses, since

$$\int_0^\infty |L_{k,u}[f(x)]|\,du = d_k\int_0^\infty u^{k-1}du\int_0^\infty \frac{t^k}{(t+u)^{2k}}|d\alpha(t)| \leqq \int_0^\infty |d\alpha(t)|.$$

THEOREM 6.2. *If* (1.1) *converges, and if $\psi(t)$ is continuous in* $(0, R)$, *then*

$$\lim_{k\to\infty}\int_0^R \psi(t)L_{k,t}[f(x)]dt = \int_0^R \psi(t)d\alpha(t) - \psi(0)\alpha(0+).$$

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* By this we mean that $\phi(t)$ is continuous for every non-negative value of t and that $\phi(t)$ approaches a limit as t becomes infinite.

† See, for example, G. C. Evans, *The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems*, American Mathematical Society Colloquium Publications, vol. 6, 1927, p. 15. The infinite intervals in question may be transformed into finite intervals by the transformation $v=e^{-u}$.

Here, we are no longer assuming that $\alpha(t)$ is of bounded variation in $(0, \infty)$. Let δ be an arbitrary positive constant, and set

$$f(x) = \int_0^{R+\delta} \frac{d\alpha(t)}{x+t} + \int_{R+\delta}^{\infty} \frac{d\alpha(t)}{x+t} = f_1(x) + f_2(x).$$

Clearly, Theorem 6.1 is applicable to $f_1(x)$. Let $\psi_R(t)$ be continuous in the infinite interval $(0 \leqq t \leqq \infty)$, coinciding with $\psi(t)$ in $(0, R)$ and constant in (R, ∞). Then

$$\begin{aligned}\lim_{k\to\infty} \int_0^{\infty} \psi_R(t) L_{k,t}[f_1(x)]dt &= \lim_{k\to\infty} \int_0^R \psi(t) L_{k,t}[f_1(x)]dt + \lim_{k\to\infty} \psi(R) \int_R^{\infty} L_{k,t}[f_1(x)]dt \\ &= \int_0^R \psi(t) d\alpha(t) + \psi(R) \int_R^{R+\delta} d\alpha(t) - \psi(0)\alpha(0+).\end{aligned}$$

Making use of Corollary 3.12, we have

$$\begin{aligned}\lim_{k\to\infty} \int_R^{\infty} \psi_R(t) L_{k,t}[f_1(x)]dt &= \lim_{k\to\infty} \psi(R) \int_R^{\infty} L_{k,t}[f_1(x)]dt \\ &= \psi(R)[\alpha(t+R) - \alpha(R)].\end{aligned}$$

On the other hand

$$\lim_{k\to\infty} \int_0^R \psi(t) L_{k,t}[f_2(x)]dt = 0,$$

as one sees by Corollary 5.21. If we combine these results, our theorem is proved.

THEOREM 6.3. *If $\alpha(t)$ has variation $V(R)$ in the interval $0 \leqq t \leqq R$, and if* (1.1) *converges, then*

$$\lim_{k\to\infty} \int_0^R |L_{k,t}[f(x)]| \, dt = V(R) - V(0+). \tag{6.2}$$

It is sufficient to prove the theorem when $\alpha(0+) = \alpha(0) = V(0+) = 0$.

If we set

$$g(x) = \int_0^{\infty} \frac{dV(t)}{x+t},$$

it is clear that

$$|L_{k,t}[f(x)]| \leqq L_{k,t}[g(x)].$$

Hence Theorem 3.1 gives

$$\limsup_{k\to\infty} \int_0^R | L_{k,t}[f(x)] | \, dt \leqq V(R). \tag{6.3}$$

On the other hand, from Theorem 6.2 we have for any function $\psi(t)$ continuous in $(0, R)$

$$\left| \int_0^R \psi(t) d\alpha(t) \right| \leqq \max_{0\leqq t\leqq R} | \psi(t) | \liminf_{k\to\infty} \int_0^R | L_{k,t}[f(x)] | \, dt.$$

Hence the norm of the linear functional

$$\int_0^R \psi(t) d\alpha(t)$$

which is known to be $V(R)$, is at most

$$\liminf_{k\to\infty} \int_0^R | L_{k,t}[f(x)] | \, dt.$$

That is,

$$V(R) \leqq \liminf_{k\to\infty} \int_0^R | L_{k,t}[f(x)] | \, dt. \tag{6.4}$$

Inequalities (6.3) and (6.4) can not both hold unless (6.2) is true.

COROLLARY 6.31. *If $V(\infty) < \infty$ then*

$$\lim_{k\to\infty} \int_0^\infty | L_{k,t} | \, dt = V(\infty) - V(0+).$$

7. **Differentiation and integration of the inversion operator.** Without any restrictions on the integral (1.1) except that it should converge we are able to obtain inversion formulas for the successive integrals of $\alpha(t)$. The following result is seen to be a generalization of Theorem 3.1.

THEOREM 7.1. *If* (1.1) *converges, and if m is any positive integer, then*

$$\lim_{k\to\infty} \int_0^t \frac{(t-u)^m}{m!} L_{k,u}[f(x)] du = \int_0^t du_m \int_0^{u_m} du_{m-1} \int_0^{u_{m-1}} \cdots \int_0^{u_1} d\alpha(u) - \alpha(0+) \frac{t^m}{m!}.$$

The result follows at once from Theorem 6.1 with

$$\psi(u) = \frac{(t-u)^m}{m!}.$$

The successive derivatives of $\alpha(u)$ or $\phi(u)$, when they exist, may be obtained in several ways. We first prove

THEOREM 7.2. *If $\phi(t)$ is of class $C^{(n)}$ in the interval $0 \leq t < \infty$, if*

$$\int_0^\infty \frac{\phi^{(n)}(t)}{x+t}\,dt \tag{7.1}$$

converges, and if $f(x)$ is defined by (4.1), *then*

$$\lim_{k\to\infty} L_{k,t}[(-1)^n f^{(n)}(x)] = \phi^{(n)}(t) \qquad (t > 0).$$

Since (7.1) converges, we have

$$\int_0^t \phi^{(n)}(u)du = o(t) \qquad (t \to \infty).$$

Hence

$$\phi^{(j)}(t) = o(t^{n-j}) \qquad (t \to \infty; j = 0, 1, 2, \cdots, n-1), \tag{7.2}$$

so that integration by parts gives us

$$\int_0^\infty \frac{\phi^{(n)}(t)}{x+t}\,dt = -\frac{\phi^{(n-1)}(0)}{x} - \frac{\phi^{(n-2)}(0)}{x^2} - \frac{2\phi^{(n-3)}(0)}{x^3} - \cdots$$

$$- (n-1)!\frac{\phi(0)}{x^n} + n!\int_0^\infty \frac{\phi(t)}{(x+t)^{n+1}}\,dt$$

$$= (-1)^n f^{(n)}(x) - \sum_{j=1}^n (j-1)!\frac{\phi^{(n-j)}(0)}{x^j}.$$

The result now follows by use of Corollary 4.11.

COROLLARY 7.21. *If $\alpha(t)$ is of class C^n in $(0, \infty)$, if the integral*

$$\int_0^\infty \frac{\alpha^{(n)}(t)}{x+t}\,dt$$

converges, and if $f(x)$ is defined by (1.1), *then*

$$\lim_{k\to\infty} L_{k,t}[(-1)^{n-1} f^{(n-1)}(x)] = \alpha^{(n)}(t) \qquad (t > 0).$$

A more natural procedure is given in

THEOREM 7.3. *Under the conditions of Theorem* 7.2

$$\lim_{k\to\infty} \frac{d^n}{dt^n} L_{k,t}[f(x)] = \phi^{(n)}(t).$$

We have at once

$$\frac{\partial^n}{\partial t^n}\frac{t^{k-1}u^k}{(t+u)^{2k}} = (-1)^n\frac{\partial^n}{\partial u^n}\frac{t^{k-n-1}u^{k+n}}{(t+u)^{2k}},$$

so that

$$\frac{d^n}{dt^n}L_{k,t}[f(x)\ ] = (-1)^n d_k\int_0^\infty \phi(u)\frac{\partial^n}{\partial u^n}\frac{t^{k-n-1}u^{k+n}}{(t+u)^{2k}}\,du.$$

Conditions (7.2) now enable us to integrate by parts and obtain

$$\frac{d^n}{dt^n}L_{k,t}[f(x)] = \frac{1}{t^n}\int_0^\infty u^n\phi^{(n)}(u)\frac{t^{k-1}}{(t+u)^{2k}}\,du,$$

for values of k sufficiently large. But the asymptotic behavior of this integral as k becomes infinite was determined in §4. It clearly approaches the desired value.

COROLLARY 7.31. *Under the conditions of Corollary* 7.21

$$\lim_{k\to\infty}\frac{d^{n-1}}{dt^{n-1}}L_{k,t}[f(x)] = \alpha^{(n)}(t).$$

8. **A generalized Stieltjes transform.** The results of the preceding section suggest a way of inverting the general Stieltjes transform

$$F(x) = \int_0^\infty \frac{d\alpha(t)}{(x+t)^\rho},$$

where ρ is any positive number. We first prove

LEMMA 8.11. *If* $t>0,\ x>0,\ \rho>0$, *then*

$$c_k\int_0^x \frac{(x-u)^{\rho-1}}{\Gamma(\rho)}(-u)^{k-1}\frac{\partial^{2k-1}}{\partial u^{2k-1}}\left[\frac{u^k\Gamma(\rho+1)}{(u+t)^{\rho+1}}\right]du = d_k\frac{t^{k-\rho}x^{k+\rho-1}}{(x+t)^{2k}}.$$

If $0\leqq u<t$,

$$\frac{(x-u)^{\rho-1}}{\Gamma(\rho)}(-u)^{k-1}\frac{\partial^{2k-1}}{\partial u^{2k-1}}\left[\frac{u^k\Gamma(\rho+1)}{(u+t)^{\rho+1}}\right]$$
$$= \sum_{n=k-1}^{\infty}\rho\binom{n+\rho}{n}\frac{(n+k)!}{(n-k+1)!}\frac{(-1)^{n+k-1}}{t^{n+\rho+1}}(x-u)^{\rho-1}u^n.$$

If $x<t$ integration term by term is permissible, so that

$$\int_0^x \frac{(x-u)^{\rho-1}}{\Gamma(\rho)}(-u)^{k-1}\frac{\partial^{2k-1}}{\partial u^{2k-1}}\left[\frac{u^k\Gamma(\rho+1)}{(u+t)^{\rho+1}}\right]du$$

$$= \sum_{n=k-1}^{\infty} \binom{n+\rho}{n} \frac{(n+k)!}{(n-k+1)!} \frac{(-1)^{n+k-1}}{t^{n+\rho+1}} \frac{x^{n+\rho}n!}{\Gamma(n+\rho+1)}$$

$$= (2k-1)! \frac{x^{\rho+k-1}}{t^{\rho+k}} \sum_{n=0}^{\infty} \binom{n+2k-1}{n} \left(\frac{-x}{t}\right)^n$$

$$= (2k-1)! \frac{t^{k-\rho}x^{k+\rho-1}}{(x+t)^{2k}} \cdot$$

It can now be seen by analytic continuation that the formula holds for all positive x and t.

LEMMA 8.12. *If the integral*

$$\int_0^\infty \frac{d\alpha(t)}{(x+t)^\rho} \qquad (\rho > 0)$$

converges, then

(8.1) $$\alpha(t) = o(t^\rho) \qquad (t \to \infty).$$

The proof is similar to that of Theorem 1.2.

By use of these results we now prove

THEOREM 8.1. *If* $0<\rho<1$, *and if the integral*

(8.2) $$F(x) = \int_0^\infty \frac{d\alpha(t)}{(x+t)^\rho},$$

converges, then

(8.3) $$\lim_{k\to\infty} \int_0^t (t-u)^{\rho-1} L_{k,u}[F(x)]du = \alpha(t .$$

Using Lemma 8.12 we have

$$F(x) = \rho \int_0^\infty \frac{\alpha(t)}{(x+t)^{\rho+1}}\, dt.$$

Then

$$L_{k,t}[F(x)] = c_k\rho \int_0^\infty \alpha(u)(-t)^{k-1} \frac{\partial^{2k-1}}{\partial t^{2k-1}} \left[\frac{t^k}{(t+u)^{\rho+1}}\right] du.$$

By uniform convergence we have for $0<\delta<y/2$

$$\int_\delta^{y-\delta} \frac{(y-t)^{\rho-1}}{\Gamma(\rho)} L_{k,t}[F(x)]dt$$

$$= c_k\rho \int_0^\infty \alpha(u)du \int_\delta^{y-\delta} \frac{(y-t)^{\rho-1}}{\Gamma(\rho)} (-t)^{k-1} \frac{\partial^{2k-1}}{\partial t^{2k-1}} \left[\frac{t^k}{(t+u)^{\rho+1}}\right] dt.$$

We may now replace δ by zero on both sides of this equation. To justify this step it is sufficient to show that

$$\int_0^y (y-t)^{\rho-1}dt \int_0^\infty |\alpha(u)|\, t^{k-1}\left|\frac{\partial^{2k-1}}{\partial t^{2k-1}}\left[\frac{t^k}{(t+u)^{\rho+1}}\right]\right| du$$

converges. For this it is sufficient to show that the integrals

$$(8.4) \qquad \int_0^y (y-t)^{\rho-1}dt \int_0^\infty |\alpha(u)| \frac{t^{2k-p-1}}{(t+u)^{\rho+2k-p}}\, du \qquad (p = 0, 1, \cdots, k)$$

all converge. But

$$\alpha(u) = \begin{cases} O(1) & (u \to 0), \\ O(u^\rho) & (u \to \infty), \end{cases}$$

so that

$$\int_0^\infty \frac{t^{2k-p-1}}{(t+u)^{\rho+2k-p}} |\alpha(u)|\, du = O(t^{-\rho}) \qquad (t \to 0;\ p = 0, 1, \cdots, k).$$

This proves the convergence of the integrals (8.4) and hence that

$$\int_0^y \frac{(y-t)^{\rho-1}}{\Gamma(\rho)} L_{k,t}[F(x)]dt$$
$$= c_k\rho \int_0^\infty \alpha(u)du \int_0^y \frac{(y-t)^{\rho-1}}{\Gamma(\rho)} (-t)^{k-1} \frac{\partial^{2k-1}}{\partial t^{2k-1}}\left[\frac{t^k}{(t+u)^{\rho+1}}\right] dt$$
$$= \frac{d_k}{\Gamma(\rho)} \int_0^\infty \alpha(u) \frac{u^{k-\rho}y^{k+\rho-1}}{(y+u)^{2k}}\, du.$$

If we use the known asymptotic expression of this last integral ($k \to \infty$), our theorem is established.

We illustrate the theorem by an example. Take

$$\begin{cases} \alpha(t) = 1 & (t > 0), \\ \alpha(0) = 0, \end{cases}$$
$$F(x) = \frac{1}{x^\rho} \cdot$$

Then

$$L_{k,u}[F(x)] = \frac{(k-\rho)(k-\rho-1)\cdots(1-\rho)\rho(\rho+1)\cdots(\rho+k-2)}{k!(k-2)!} \frac{1}{u^\rho},$$

$$\int_0^t (t-u)^{\rho-1} L_{k,u}[F(x)]du = \frac{\Gamma(\rho+k-1)\Gamma(\rho-k+1)}{k!(k-2)!} \qquad (t > 0),$$

and the right-hand side clearly approaches unity as k becomes infinite. This example shows in particular that the restriction $\rho<1$ in Theorem 8.1 was essential, for if $\rho>1$, the left-hand side of (8.3) need not converge.

If ρ is not less than unity we must proceed differently. In order to treat this case we introduce a new operator.

DEFINITION 8.1. *An operator* $L^{\rho}_{k,t}[f(x)]$ *is defined by the equation*

$$L^{\rho}_{k,t}[f(x)] = (-1)^k\Gamma(\rho)c_k[t^{2k-1}\{f^{(k)}(t)\}^{(-\rho)}]^{(k)}.$$

In this definition ρ is any positive number, k is a positive integer greater than ρ. The notation is understood to mean

$$\{f^{(k)}(t)\}^{(-\rho)} = \int_t^\infty \frac{(u-t)^{\rho-1}}{\Gamma(\rho)} f^{(k)}(u)du$$

if ρ is not an integer. If ρ is an integer

$$\{f^{(k)}(t)\}^{(-\rho)} = (-1)^\rho f^{(k-\rho)}(t).$$

It must not be supposed that for fractional ρ the function $\{f^{(k)}(t)\}^{(-\rho)}$ is the fractional derivative of $f(t)$ of order $k-\rho$, defined for $0<\rho<1$ by

$$\frac{d^k}{dt^k}\int_t^\infty \frac{(u-t)^{\rho-1}}{\Gamma(\rho)} f(u)du.$$

For, this integral need not exist in the present case. For example, if $f(t)=t^{-1/2}$, the integral does not exist if $\rho=1/2$. Yet the operator $L^{1/2}_{k,t}[f(x)]$ clearly exists for all integers k not less than unity.

We prove next

THEOREM 8.2. *If the integral* (8.2) *converges, then*

$$\lim_{k\to\infty}\int_0^t L^{\rho}_{k,u}[F(x)]du = \alpha(t) - \alpha(0+) \qquad (t>0).$$

We first prove that for k sufficiently large

$$\{f^{(k)}(x)\}^{(-\rho)} = (-1)^k\frac{(k-1)!}{\Gamma(\rho)}\int_0^\infty \frac{d\alpha(t)}{(x+t)^k}. \tag{8.5}$$

To see this we have

$$\begin{aligned}(-1)^kF^{(k)}(x) &= \frac{\Gamma(\rho+k)}{\Gamma(\rho)}\int_0^\infty \frac{d\alpha(t)}{(x+t)^{\rho+k}} \\ &= \frac{\Gamma(\rho+k+1)}{\Gamma(\rho)}\int_0^\infty \frac{\alpha(t)}{(x+t)^{\rho+k+1}}dt,\end{aligned} \tag{8.6}$$

$$\int_x^\infty \frac{(t-x)^{\rho-1}}{\Gamma(\rho)} F^{(k)}(t)dt$$

$$= (-1)^k \frac{\Gamma(\rho+k+1)}{\Gamma(\rho)} \int_x^\infty \frac{(t-x)^{\rho-1}}{\Gamma(\rho)} dt \int_0^\infty \frac{\alpha(u)}{(t+u)^{\rho+k+1}} du.$$

If $\delta>0$, $R>0$, $x+\delta<R$, the uniform convergence of the integral (8.6) shows us that

$$\int_x^\infty \frac{(t-x)^{\rho-1}}{\Gamma(\rho)} F^{(k)}(t)dt$$

$$= \lim_{\substack{\delta\to 0\\ R\to\infty}} (-1)^k \frac{\Gamma(\rho+k+1)}{\Gamma(\rho)} \int_0^\infty \frac{\alpha(u)}{\Gamma(\rho)} du \int_{x+\delta}^R \frac{(t-x)^{\rho-1}}{(t+u)^{\rho+k+1}} dt$$

$$= (-1)^k \frac{\Gamma(\rho+k+1)}{\Gamma(\rho)} \int_0^\infty \frac{\alpha(u)}{\Gamma(\rho)} du \int_x^\infty \frac{(t-x)^{\rho-1}}{(t+u)^{\rho+k+1}} dt$$

$$= (-1)^k \frac{\Gamma(k+1)}{\Gamma(\rho)} \int_0^\infty \frac{\alpha(u)}{(x+u)^{k+1}} du \tag{8.7}$$

provided the integral (8.7) converges absolutely. This it clearly does for $k>\rho$ by virtue of the relation (8.1). An integration by parts now gives (8.5). Then we obtain

$$\int_0^t L^\rho_{k,u}[F(x)]du = d_k t^k \int_0^\infty \frac{y^{k-1}\alpha(y)}{(t+y)^{2k}} dy - \frac{k-1}{k}\alpha(0+)$$

precisely as in the proof of Theorem 3.1. The theorem is now established by use of Corollary 3.13.

In a similar way we may obtain a generalization of Theorem 4.1.

THEOREM 8.3. *If $\phi(t)$ is integrable in* $(0, R)$ *for every positive* R, *and if* $F(x)$ *is defined by the convergent integral*

$$F(x) = \int_0^\infty \frac{\phi(t)}{(x+t)^\rho} dt \qquad (\rho>0),$$

then

$$\lim_{k\to\infty} L^\rho_{k,t}[F(x)] = \phi(t)$$

at every point t of the Lebesgue set for $\phi(u)$.

Let us illustrate these theorems by use of the same example as we used for illustration of Theorem 8.1.

$$F(x) = \frac{1}{x^\rho} = \int_0^\infty \frac{d\alpha(t)}{(x+t)^\rho} = \int_0^\infty \frac{\rho dt}{(x+t)^{\rho+1}} = \int_0^\infty \frac{\rho(\rho+1)t}{(x+t)^{\rho+2}}\, dt.$$

Direct application of Definition 8.1 gives us

$$L_{k,t}^{\rho}[F(x)] = 0,$$
$$L_{k,t}^{\rho+1}[F(x)] = \rho,$$
$$L_{k,t}^{\rho+2}[F(x)] = \rho(\rho+1)\frac{k+1}{k-2}\,t.$$

In each case the appropriate limit process gives the result predicted by Theorems 8.2 and 8.3.

9. **Uniqueness.** Of fundamental importance in later work will be the uniqueness theorem for the operator $L_{k,t}$. As a preliminary result we establish

THEOREM 9.1. *Let $f(t)$, $g(t)$ be functions of class C^{2k} in $0<t<\infty$, and let*

$$(9.1)\qquad \lim\ [t^{2k-1}f^{(k)}(t)]^{(k-p)}g^{(p-1)}(t) = 0 \qquad (t\to 0,\ t\to\infty;\ p = 1, 2, \cdots, k),$$

$$(9.2)\qquad \lim\ [t^{2k-1}g^{(k)}(t)]^{(k-p)}f^{(p-1)}(t) = 0 \qquad (t\to 0,\ t\to\infty;\ p = 1, 2, \cdots, k).$$

Then

$$\int_0^\infty [t^{2k-1}f^{(k)}(t)]^{(k)}g(t)dt = \int_0^\infty [t^{2k-1}g^{(k)}(t)]^{(k)}f(t)dt,$$

if either integral exists.

To verify this one has only to integrate successively by parts. Conditions (9.1) and (9.2) guarantee that at each stage the integrated part vanishes.

THEOREM 9.2. *If $f(x)$ is of class C^{2k-1} in the interval $0<x<\infty$, and if*

$$(9.3)\qquad \lim\ [t^{2k-1}f^{(k-1)}(t)]^{(k-p)}(t+a)^{-p} = 0 \qquad (t\to 0,\ t\to\infty;\ p = 1, 2, \cdots, k;\ a>0),$$

$$(9.4)\qquad \lim\ [t^{2k-1}(t+a)^{-k-1}]^{(k-p-2)}f^{(p)}(t) = 0 \qquad (t\to 0,\ t\to\infty;\ p = 0, 1, \cdots, k-2),$$

$$(9.5)\qquad \begin{aligned} f(t) &= O(t^\mu) && (t\to\infty,\ \mu < k-1),\\ f(t) &= O(t^{-\nu}) && (t\to 0,\ \nu < k+1), \end{aligned}$$

then

$$\int_0^\infty \frac{L_{k,t}[f(x)]}{t+a}\, dt = \int_0^\infty L_{k,a}\left[\frac{1}{x+t}\right] f(t)dt.$$

This result may be obtained at once from Theorem 9.1. For in that theorem replace $g(t)$ by $(t+a)^{-1}$ and $f(t)$ by

$$f_1(t) = \int_1^t f(u)du.$$

We obtain

$$\int_0^\infty \frac{L_{k,t}[f(x)]}{t+a}\,dt = (-1)^{2k-1}c_k\int_0^\infty \left[\frac{t^{2k-1}k!}{(t+a)^{k+1}}\right]^{(k)} f_1(t)dt. \tag{9.6}$$

Integrating the right-hand member by parts gives

$$\begin{aligned}\int_0^\infty \frac{L_{k,t}[f(x)]}{t+a}\,dt &= c_k\int_0^\infty \left[\frac{t^{2k-1}k!}{(t+a)^{k+1}}\right]^{(k-1)} f(t)dt,\\ &= d_k\int_0^\infty \frac{a^{k-1}t^k}{(t+a)^{2k}}\,f(t)dt,\\ &= \int_0^\infty L_{k,a}\left[\frac{1}{x+t}\right] f(t)dt.\end{aligned} \tag{9.7}$$

It remains only to verify that the integral (9.7) converges. It clearly does by virtue of (9.5).

We come now to the uniqueness theorem.

THEOREM 9.3. *If $f(x)$ is of class C^∞ in $(0<x<\infty)$, if (9.3), (9.4) hold for $k=2, 3, 4, \cdots$, and if (9.5) holds for some positive μ* and *ν which are independent of k, then*

$$f(a) = \lim_{k\to\infty}\int_0^\infty \frac{L_{k,t}[f(x)]}{t+a}\,dt \qquad (a>0). \tag{9.8}$$

The proof follows at once by use of (3.2).

Note that if $f_1(x)$ and $f_2(x)$ are two functions satisfying the condition of Theorem 9.3 and such that

$$L_{k,t}[f_1(x)] = L_{k,t}[f_2(x)]$$

for all sufficiently large integers k, then

$$f_1(x) = f_2(x) \qquad (0<x<\infty).$$

It is for this reason that the result may be regarded as a uniqueness theorem for the operator $L_{k,t}[f(x)]$.

10. **Sufficient conditions for uniqueness.** Conditions (9.3), (9.4), and (9.5) are sufficient for the application of Theorem 9.3. In this form, however, it may be difficult to determine, in any given case, whether a function satisfies

them or not. It is the purpose of the present section to replace them by conditions more easily applied.

THEOREM 10.1. *If*

$$(10.1)\qquad \begin{aligned} f^{(k)}(x) &= o\left(\frac{1}{x^{k+1}}\right) \qquad (x\to 0;\ k=0,1,2,\cdots),\\ f^{(k)}(x) &= o\left(\frac{1}{x^{k}}\right) \qquad (x\to \infty;\ k=0,1,2,\cdots),\end{aligned}$$

then equation (9.8) *is true.*

For, if one expands (9.3) and (9.4) by Leibniz's rule, one sees that (9.3), (9.4), and (9.5) are true for all positive integers k by virtue of the relations (10.1). Note that no positive or negative integral power of x satisfies (10.1). For such functions $f(x)$ the right-hand side of (9.8) is zero.

For use in the proof of our next result we establish

LEMMA 10.21. *If k is a positive integer, and if*

$$f^{(k-1)}(t) = O\left(\frac{1}{t^k}\right) \qquad (t\to 0),$$

then

$$\int_0^t u^{2k+1}f^{(k)}(u)du = O(t^{k+1}) \qquad (t\to 0).$$

For, integration by parts gives for $\epsilon>0$

$$\int_\epsilon^t u^{2k+1}f^{(k)}(u)du = t^{2k+1}f^{(k-1)}(t) - \epsilon^{2k+1}f^{(k-1)}(\epsilon) - (2k+1)\int_\epsilon^t u^{2k}f^{(k-1)}(u)du.$$

Making use of our hypothesis regarded $f^{(k-1)}(t)$, we obtain

$$\begin{aligned}\int_0^t u^{2k+1}f^{(k)}(u)du &= t^{2k+1}f^{(k-1)}(t) - (2k+1)\int_0^t u^{2k}f^{(k-1)}(u)du\\ &= O(t^{k+1}) \qquad (t\to 0).\end{aligned}$$

LEMMA 10.22. *If $f(x)$ is of class C^∞ in the interval $0<x<\infty$, and if the limits*

$$\lim_{\epsilon\to 0+}\int_\epsilon^1 [u^{2k-1}f^{(k-1)}(u)]^{(k)}du \qquad (k=1,2,3,\cdots)$$

exist, then

$$(-1)^k f^{(k)}(x) \sim \frac{Ak!}{x^{k+1}} \qquad (x\to 0+),$$

for a suitable constant A.

The hypothesis for $k=1$ assures us that the Cauchy value of the integral

$$\int_0^1 [uf(u)]'du$$

exists, and hence that there exists a constant A for which

$$tf(t) \sim A \qquad (t \to 0).$$

In particular

$$f(t) = O\left(\frac{1}{t}\right) \qquad (t \to 0).$$

We now proceed by induction and assume that

$$f^{(p)}(t) = O\left(\frac{1}{t^{p+1}}\right) \qquad (t \to 0;\ p = 0, 1, \cdots, k-1).$$

Since

$$\lim_{\epsilon \to 0+} \int_\epsilon^1 [u^{2k+1}f^{(k)}(u)]^{(k+1)}du$$

exists, it follows that

$$[u^{2k+1}f^{(k)}(u)]^{(k)} = O(1) \qquad (u \to 0).$$

Hence

$$\int_0^t [u^{2k+1}f^{(k)}(u)]^{(k)}du = O(t) \qquad (t \to 0),$$

$$[t^{2k+1}f^{(k)}(t)]^{(k-1)} - c_1 = O(t) \qquad (t \to 0),$$

for a suitable constant c_1. By successive integrations we have

$$\int_0^t u^{2k+1}f^{(k)}(u)du + P_k(t) = O(t^{k+1}) \qquad (t \to 0),$$

where $P_k(t)$ is a polynomial of degree at most equal to k. By Lemma 10.21

$$P_k(t) = O(t^{k+1}) \qquad (t \to 0).$$

But this is impossible unless $P_k(t)$ is identically zero. However,

$$\int_0^t [u^{2k+1}f^{(k)}(u)]'du + P_k'(t) = O(t^k),$$

$$t^{2k+1}f^{(k)}(t) = O(t^k),$$

$$f^{(k)}(t) = O\left(\frac{1}{t^{k+1}}\right) \qquad (t \to 0).$$

That is, this last relation must hold for all positive integers k. By use of a theorem of Hardy and Littlewood* the proof of the lemma is completed.

We can now prove

THEOREM 10.2. *If for each positive integer* k

$$\int_0^R L_{k,t}[f(x)]dt = O(R) \qquad (R\to\infty), \tag{10.2}$$

then

$$(-1)^k f^{(k)}(x) \sim \frac{A\,k!}{x^{k+1}} \qquad (x\to 0;\ k=0,1,2,\cdots), \tag{10.3}$$

$$f^{(k)}(x) = O\left(\frac{1}{x^k}\right) \qquad (x\to\infty;\ k=0,1,2,\cdots), \tag{10.4}$$

where

$$A = \lim_{x\to 0+} xf(x). \tag{10.5}$$

The conclusions (10.3) and (10.5) follow at once from Lemma 10.22. To prove (10.4) we have

$$\int_0^R [t^{2k-1}f^{(k-1)}(t)]^{(k)}dt = O(R) \qquad (R\to\infty),$$

for each positive integer k. This shows that

$$[t^{2k-1}f^{(k-1)}(t)]^{(k-1)} = O(t) \qquad (t\to\infty),$$

$$t^{2k-1}f^{(k-1)}(t) = O(t^k) \qquad (t\to\infty),$$

from which (10.4) follows at once.

Our next result is

THEOREM 10.3. *If* $f(x)$ *satisfies* (10.2) *for each positive integer* k, *and if*

$$\lim_{x\to\infty} f(x) = 0, \tag{10.6}$$

then

$$f(a) = \lim_{k\to\infty}\int_0^\infty \frac{L_{k,t}[f(x)]}{t+a}dt + \frac{A}{a} \qquad (a>0), \tag{10.7}$$

where

$$A = \lim_{x\to 0+} xf(x).$$

* See, for example, E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1929, p. 58.

For, set

$$g(x) = f(x) - \frac{A}{x}\cdot$$

Then

$$g^{(k)}(x) = o\left(\frac{1}{x^{k+1}}\right) \qquad (x \to 0;\ k = 0, 1, 2, \cdots),$$

by Theorem 10.2. The same theorem shows that (10.4) is true for $g(x)$. This combined with (10.6) implies

$$g^{(k)}(x) = o\left(\frac{1}{x^{k}}\right) \qquad (x \to \infty;\ k = 0, 1, 2, \cdots).$$

Hence, by Theorem 10.1,

$$g(a) = \lim_{k\to\infty} \int_0^\infty \frac{L_{k,t}[g(x)]}{a+t}\,dt \qquad (a > 0).$$

Since

$$L_{k,t}[f(x)] = L_{k,t}[g(x)],$$

we see that (10.7) follows at once.

Corollary 10.31. *If the functions* $L_{k,t}[f(x)]$, $(k=1, 2, \cdots)$, *are all bounded, then* (10.6) *implies* (10.7).

Corollary 10.32. *If*

$$(10.8) \qquad \int_0^\infty | L_{k,t}[f(x)] |^p dt < \infty \qquad (k = 1, 2, \cdots;\ p \geqq 1),$$

then (10.7) *holds.*

For, if $p=1$, then

$$\int_0^\infty | [tf(t)]' |\, dt < \infty,$$

so that (10.6) must hold. Clearly (10.8) implies (10.2) in this case. If $p>1$, Hölder's inequality gives

$$(10.9) \qquad \int_0^R | L_{k,t}[f(x)] |\, dt \leqq \left[\int_0^\infty | L_{k,t}[f(x)] |^p dt\right]^{1/p} R^{(p-1)/p},$$

so that (10.2) is satisfied. For $k=1$, (10.9) becomes

$$f(t) = O\left(\frac{1}{t^{1/p}}\right) \qquad (t \to \infty).$$

Hence (10.6) is satisfied. That is, Theorem 10.3 is applicable.

11. $\alpha(t)$ **of bounded variation.** Here we develop a necessary and sufficient condition that the equation (1.1) should have a solution $\alpha(t)$ of bounded variation in the infinite interval $(0, \infty)$.

THEOREM 11.1. *A necessary and sufficient condition that $f(x)$ should have the representation* (1.1) *with $\alpha(t)$ of bounded variation in* $(0, \infty)$ *is that*

$$\int_0^\infty |L_{k,t}[f(x)]|\,dt < M \qquad (k = 1, 2, \cdots), \tag{11.1}$$

where M is some constant independent of k.

To prove the necessity we have

$$L_{k,t}[f(x)] = d_k \int_0^\infty \frac{u^k t^{k-1}}{(t+u)^{2k}}\,d\alpha(u),$$

where

$$\int_0^\infty |d\alpha(u)| < \infty.$$

Then

$$\begin{aligned}\int_0^\infty |L_{k,t}[f(x)]|\,dt &\leqq d_k \int_0^\infty t^{k-1}dt \int_0^\infty \frac{u^k}{(t+u)^{2k}} |d\alpha(u)| \\ &= d_k \int_0^\infty u^k |d\alpha(u)| \int_0^\infty \frac{t^{k-1}}{(t+u)^{2k}}\,dt \\ &= \frac{k-1}{k}\int_0^\infty |d\alpha(u)| \leqq \int_0^\infty |d\alpha(u)|.\end{aligned}$$

This proves the necessity.

For the sufficiency, we have by Corollary 10.32

$$f(a) = \lim_{k\to\infty} \int_0^\infty \frac{L_{k,t}[f(x)]}{t+a}\,dt + \frac{A}{a} \qquad (a > 0),$$

$$A = \lim_{x\to 0+} xf(x).$$

By a theorem of Helly* we can pick from the set of functions

$$\alpha_k(t) = \int_0^t L_{k,u}[f(x)]\,du \qquad (k = 1, 2, \cdots)$$

* E. Helly, *Über lineare Funktionaloperationen*, Wiener Sitzungsberichte, vol. 121 (1921), p. 265.

a subset $\alpha_{k_i}(t)$ which approaches a function $\alpha^*(t)$ of bounded variation in the interval $(0, \infty)$. Then

$$f(a) = \lim_{i\to\infty} \int_0^\infty \frac{d\alpha_{k_i}(t)}{t+a} + \frac{A}{a} \qquad (a > 0)$$

By the Helly-Bray† theorem we may take the limit under the integral sign and obtain

$$f(a) = \int_0^\infty \frac{d\alpha^*(t)}{t+a} + \frac{A}{a}$$
$$= \int_0^\infty \frac{d\alpha(t)}{t+a} \qquad (a > 0),$$

where $\alpha(t)$ vanishes at the origin and differs from $\alpha^*(t)$ by the constant A for positive values of t. This completes the proof of the theorem.

12. $\alpha(t)$ **non-decreasing.** Let us introduce

DEFINITION 12.1. *A function $f(x)$ satisfies Property* A *if and only if*

$$L_{k,t}[f(x)] \geqq 0 \qquad (t > 0;\ k = 0, 1, 2, \cdots).$$

Clearly this is equivalent to

$$f(x) \geqq 0, \quad (-1)^{k-1}[x^{2k-1}f^{(k-1)}(x)]^{(k)} \geqq 0 \qquad (x > 0;\ k = 1, 2, \cdots),$$

or to

$$f(x) \geqq 0, \qquad (-1)^{k-1}[x^k f(x)]^{(2k-1)} \geqq 0 \qquad (x > 0;\ k = 1, 2, \cdots).$$

In the proof of our result we shall need

LEMMA 12.11. *If $\phi(x)$ is of class C^1 in $0<x\leqq 1$ and if $\phi'(x)$ is bounded on one side in that interval, then $-\phi(x)$ is bounded on the same side there.*

The proof is obvious.

THEOREM 12.1. *If $f(x)$ has Property* A, *then the relations* (10.3) *and* (10.5) *hold.*

Since $xf(x)$ is a positive increasing function, it follows that the constant A of (10.5) is well defined. It will be sufficient to prove that

$$f^{(p)}(x) = O\left(\frac{1}{x^{p+1}}\right) \qquad (x \to 0;\ p = 0, 1, 2, \cdots).$$

Since this has been proved for $k=0$, we may proceed by induction. Let us

† See, for example, G. C. Evans, *The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems*, American Mathematical Society Colloquium Publications, vol. 6, 1927, p. 15.

grant then that these relations hold for $p=0, 1, 2, \cdots, k-2$. By hypothesis

$$(-1)^{k-1}[x^k f(x)]^{(2k-1)} \geqq 0 \qquad (x > 0).$$

By Lemma 12.11

$$[x^k f(x)]^{(k-1)} < M \qquad (0 < x \leqq 1)$$

for a suitable constant M. Also

$$(-1)^{k-2}[x^{k-1} f(x)]^{(2k-3)} \geqq 0$$

from which we deduce in the same way that

$$[x^{k-1} f(x)]^{(k-1)} > N \qquad (0 < x \leqq 1)$$

for a suitable constant N. But

$$[x^k f(x)]^{(k-1)} = x[x^{k-1} f(x)]^{(k-1)} + (k-1)[x^{k-1} f(x)]^{(k-2)}.$$

Since the second term on the right-hand side is $O(1)$ by our assumption (10.6), it follows that

$$Nx + O(1) < [x^k f(x)]^{(k-1)} < M \qquad (0 < x \leqq 1).$$

Hence

$$[x^k f(x)]^{(k-1)} = O(1) \qquad (x \to 0).$$

Expanding by Leibniz's rule we see that

$$f^{(k-1)}(x) = O\left(\frac{1}{x^k}\right),$$

so that the induction is complete. Hence (10.3) is established.

THEOREM 12.2. *If $f(x)$ has Property* A, *then there exist constants $A_0, A_1, \cdots$ such that*

$$(12.1) \qquad (-1)^k f^{(k)}(x) \geqq \frac{A_k}{x^{k+1}} \qquad (k = 0, 1, 2, \cdots ; 0 < x < \infty).$$

Since

$$(-1)^{k-1}[x^{2k-1} f^{(k-1)}(x)]^{(k)} \geqq 0,$$

it follows by successive integration that

$$(-1)^{k-1}[x^{2k-1} f^{(k-1)}(x)] \geqq A_{k-1} x^{k-1} \qquad (1 \leqq x < \infty).$$

By Theorem 12.1 a similar inequality holds in the interval $(0<x\leqq 1)$, so that (12.1) follows at once.

THEOREM 12.3. *If $f(x)$ has Property* A *and if $xf(x)$ approaches a limit as k becomes infinite, then*

$$\int_0^\infty L_{k,t}[f(x)]dt = \frac{k-1}{k}\left[\lim_{x\to\infty} xf(x) - \lim_{x\to 0} xf(x)\right].$$

For, if

$$f(x) \sim \frac{B}{x} \qquad (x \to \infty),$$

then Theorem 12.2 with the addition of the Hardy-Littlewood result referred to earlier shows us that

$$(12.2) \qquad (-1)^k f^{(k)}(x) \sim \frac{Bk!}{x^{k+1}} \qquad (x \to \infty).$$

It is easily seen that the relations (12.2) imply that

$$\lim_{t\to\infty} (-1)^{k-1}\left[t^{2k-1}f^{(k-1)}(t)\right]^{(k-1)} = (k-1)!(k-1)!B.$$

There is of course a similar result for $t \to 0$ following from the relations (10.3) which are necessary consequences of Property A. Hence

$$\int_0^\infty L_{k,t}[f(x)]dt = c_k \int_0^\infty (-1)^{k-1}\left[t^{2k-1}f^{(k-1)}(t)\right]^{(k)}dt$$
$$= \frac{(k-1)!(k-1)!}{k!(k-2)!}[B - A].$$

This proves the theorem. It is to be noted that the existence of B added to our hypothesis in this theorem is not a consequence of Property A, as one sees by the examples $f(x) = 1$ and $f(x) = (x)^{-1/2}$. Both satisfy the property but in each case B fails to exist. We can now treat the case† of bounded non-decreasing functions $\alpha(t)$.

THEOREM 12.4. *A necessary and sufficient condition that $f(x)$ should have the form* (1.1) *with $\alpha(t)$ bounded non-decreasing is that $f(x)$ should have the Property* A *and that $xf(x)$ should approach a limit as x becomes infinite. Further,*

$$(12.3) \qquad \alpha(\infty) - \alpha(0+) = \lim_{k\to\infty}\int_0^\infty L_{k,t}[f(x)]dt.$$

The necessity of Property A follows from an inspection of the relations

† The author treated this case by another method in an earlier paper: D. V. Widder, *A classification of generating functions*, these Transactions, vol. 39 (1936), p. 244.

$$L_{0,t}[f(x)] = f(t),$$

$$L_{k,t}[f(x)] = d_k t^{k-1} \int_0^\infty \frac{u^k}{(t+u)^{2k}} d\alpha(u) \qquad (k = 1, 2, \cdots).$$

Moreover, it is obvious that

$$\lim_{x\to\infty} xf(x) = \lim_{x\to\infty} \int_0^\infty \frac{x d\alpha(t)}{x+t} = \alpha(\infty).$$

To prove the sufficiency we first appeal to Theorem 12.3. This shows that

$$\int_0^\infty L_{k,t}[f(x)]dt \leqq B - A \qquad (k = 1, 2, \cdots),$$

where A, B are defined as in the proof of Theorem 12.3. Hence (11.1) is satisfied, and, by Theorem 11.1, $f(x)$ has the representation (1.1) with $\alpha(t)$ of bounded variation in $(0, \infty)$. To show that $\alpha(t)$ is non-decreasing we now appeal to Theorem 3.1. Clearly, on the assumption of Property A, the functions

$$\int_0^t L_{k,u}[f(x)]du \qquad (k = 1, 2, \cdots)$$

are non-decreasing functions of t.

Finally, (12.3) is a direct result of Corollary 6.31.

We turn next to the case of unbounded non-decreasing functions $\alpha(t)$. For the discussion of this case we need

LEMMA 12.51. *If $f(x)$ satisfies Property* A, *then it approaches a limit as x becomes infinite.*

For, since

$$-[u^2 f(u)]^{(3)} \geqq 0 \qquad (u > 0),$$

we have for $0<y<x$ by successive integrations

$$-x^2 f(x) + y^2 f(y) + (x-y)[y^2 f(y)]' + \frac{(x-y)^2}{2}[y^2 f(y)]'' \geqq 0,$$

whence

$$\limsup_{x\to\infty} f(x) = E \leqq \tfrac{1}{2}[y^2 f(y)]'' \qquad (y > 0).$$

Successive integration of the inequality

$$[u^2 f(u)]'' - 2E \geqq 0$$

gives

$$x^2f(x) - y^2f(y) - (x - y)[y^2f(y)]' - E(x - y)^2 \geqq 0 \qquad (0 < y < x).$$

Hence

$$\liminf_{x\to\infty} f(x) \geqq E,$$

or

$$\lim_{x\to\infty} f(x) = E \geqq 0. \tag{12.4}$$

LEMMA 12.52. *If $f(x)$ satisfies Property* A, *then for each non-negative integer k the function $[x^kf(x)]^{(k)}$ is completely monotonic for $x>0$,*

$$(-1)^n[x^kf(x)]^{(k+n)} \geqq 0 \qquad (n = 0, 1, 2, \cdots).$$

By (12.4) and (12.1) it follows from the Hardy-Littlewood result quoted earlier that

$$f^{(k)}(x) = o\left(\frac{1}{x^k}\right) \qquad (k = 1, 2, \cdots; x \to \infty).$$

Since

$$(-1)^{k-1}[x^kf(x)]^{(2k-1)} \geqq 0 \qquad (x > 0),$$

and since

$$\lim_{x\to\infty} [x^kf(x)]^{(n)} = 0 \quad (n = k+1, k+2, \cdots),$$

it follows by successive integrations to infinity that

$$(-1)^n[x^kf(x)]^{(k+n)} \geqq 0 \qquad (n = 1, 2, \cdots, k-1; x > 0), \tag{12.5}$$

$$[x^kf(x)]^{(k)} \geqq E \geqq 0. \tag{12.6}$$

It remains to show that (12.5) holds for $n=k, k+1, \cdots$.

Let r be a positive integer, and replace $f(x)$ by $x^rf(x)$ in Lemma 3.11. We obtain

$$[x^{2k-1}\{x^rf(x)\}^{(k-1)}]^{(k)} = x^{k-1}[x^{k+r}f(x)]^{(2k-1)}.$$

Hence by (12.5) and (12.6)

$$(-1)^{k-r-1}[x^{k+r}f(x)]^{(2k-1)} \geqq 0 \qquad (k \geqq r+1),$$

$$(-1)^{k-r-1}[x^{2k-1}\{x^rf(x)\}^{(k-1)}]^{(k)} \geqq 0 \qquad (k \geqq r+1). \tag{12.7}$$

But

$$\lim_{x\to 0+} [x^{2k-1}\{x^rf(x)\}^{(k-1)}]^{(p)} = 0 \qquad (p = 0, 1, 2, \cdots, k-1; r > 0),$$

$$\lim_{x\to 0+} (-1)^{k-1}[x^{2k-1}f^{(k-1)}(x)]^{(k-1)} = (k-1)!(k-1)!A \geqq 0.$$

Hence successive integrations of (12.7) from zero give

$$(-1)^{k-r-1}[x^r f(x)]^{(k-1)} \geqq 0 \qquad (k \geqq r+1),$$

and this completes the proof of the lemma.

LEMMA 12.53. *If $f(x)$ has Property* A, *and if $\delta>0$, then $f(x+\delta)$ has the same property.*

For,

$$(12.8)\qquad \begin{aligned}&(-1)^{k-1}[x^k f(x+\delta)]^{(2k-1)}\\ &\qquad = \sum_{p=0}^{k}\binom{k}{p}(-1)^{p+k-1}[(x+\delta)^{k-p}f(x+\delta)]^{(2k-1)}.\end{aligned}$$

By Lemma 12.52

$$(-1)^{k+p-1}[(x+\delta)^{k-p}f(x+\delta)]^{(2k-1)} \geqq 0 \qquad (p=0,1,\cdots,k),$$

so that every term in the sum (12.8) is non-negative.

LEMMA 12.54. *If $f(x)$ has Property* A *and if $\delta>0$, then*

$$F(x) = \frac{f(x+\delta)-f(\delta)}{-x}$$

has the property and

$$(12.9)\qquad \lim_{x\to\infty} xF(x) = f(\delta) - f(\infty).$$

For

$$(-1)^{k-1}[x^k F(x)]^{(2k-1)} = (-1)^k[x^{k-1}f(x+\delta)]^{(2k-1)} \qquad (k=1,2,\cdots).$$

But the right-hand side is non-negative by Lemmas 12.52 and 12.53. Also

$$F(x) = \frac{f(x+\delta)-f(\delta)}{-x} = -f'(\xi) \geqq 0 \qquad (\delta<\xi<\delta+x),$$

so that $F(x)$ has Property A. By Lemma 12.51 we deduce (12.9).

By use of this last result we can now prove the main result of the section.

THEOREM 12.5. *Property* A *for the function $f(x)$ is necessary and sufficient that it should have the form*

$$(12.10)\qquad f(x) = E + \int_0^\infty \frac{d\alpha(t)}{x+t},$$

where $\alpha(t)$ is non-decreasing and $E\geqq 0$.

The proof of the necessity is made as in Theorem 12.4.

For the sufficiency we have at once by Lemma 12.54 and Theorem 12.4 for a given positive δ

$$F(x) = \frac{f(x+\delta) - f(\delta)}{-x} = \int_0^\infty \frac{d\beta(t)}{x+t},$$

where $\beta(t)$ is non-decreasing and bounded. In fact

$$\beta(0) = 0, \qquad \beta(\infty) = f(\delta) - f(\infty).$$

But $\beta(t)$ must be constantly zero in $(0, \delta)$, for otherwise it would have a point of increase there and by Theorem 1.7, $f(x)$ would have a singularity for some positive x. But, by Lemma 12.52, $f(x)$ is completely monotonic and hence analytic for $x>0$. Hence

$$f(x+\delta) = f(\delta) - \int_0^\infty \frac{x}{x+t}\,d\beta(t) = f(\infty) + \int_0^\infty d\beta(t) - \int_0^\infty \frac{x}{x+t}\,d\beta(t)$$

$$= f(\infty) + \int_\delta^\infty \frac{t}{x+t}\,d\beta(t) = f(\infty) + \int_0^\infty \frac{t+\delta}{x+t+\delta}\,d\beta(t+\delta),$$

$$f(x) = f(\infty) + \int_0^\infty \frac{(t+\delta)}{x+t}\,d\beta(t+\delta)$$

$$= f(\infty) + \int_0^\infty \frac{d\alpha(t)}{x+t} \qquad (x > \delta), \tag{12.11}$$

where

$$\alpha(t) = \int_0^t (u+\delta)\,d\beta(u+\delta).$$

Clearly $\alpha(t)$ is non-decreasing. It is independent of δ by Theorem 1.6. Hence (12.11) holds for all $x>0$, and our theorem is proved.

13. **$\phi(t)$ of class L^p, $p>1$.** In this section we deduce conditions on $f(x)$ which will insure its representation in the form (4.1) with

$$\int_0^\infty |\phi(t)|^p dt < \infty \tag{13.1}$$

for some constant $p>1$. The result is

THEOREM 13.1. *A necessary and sufficient condition that $f(x)$ should have the form* (4.1) *with $\phi(t)$ satisfying* (13.1) *is that*

$$\int_0^\infty |L_{k,t}[f(x)]|^p dt < M \qquad (k = 1, 2, \cdots), \tag{13.2}$$

$$\lim_{x\to 0+} xf(x) = 0, \tag{13.3}$$

where M is some constant independent of k.

For the necessity we have from §4

$$L_{k,t}[f(x)] = d_k t^{k-1} \int_0^\infty \frac{u^k}{(t+u)^{2k}} \phi(u) du \qquad (k = 1, 2, \cdots).$$

These integrals all converge absolutely since

$$\int_0^\infty \frac{u^k}{(t+u)^{2k}} |\phi(u)| du \leqq \left[\int_0^\infty |\phi(u)|^p du \right]^{1/p} \left[\frac{\Gamma(qk+1)\Gamma(qk-1)}{t^{qk-1}\Gamma(2qk)} \right]^{1/q},$$

$$\frac{1}{p} + \frac{1}{q} = 1,$$

by Hölder's inequality. We also have for $k > 1$

$$|L_{k,t}[f(x)]|^p \leqq d_k t^{k-1} \int_0^\infty \frac{u^k}{(t+u)^{2k}} |\phi(u)|^p du \left[d_k \int_0^\infty \frac{t^{k-1} u^k}{(t+u)^{2k}} du \right]^{p/q},$$

$$\int_0^\infty |L_{k,t}[f(x)]|^p dt \leqq d_k \int_0^\infty t^{k-1} dt \int_0^\infty \frac{u^k}{(t+u)^{2k}} |\phi(u)|^p du \tag{13.4}$$

$$= \frac{k-1}{k} \int_0^\infty |\phi(u)|^p du = \int_0^\infty |\phi(u)|^p du.$$

For $k = 1$, this argument fails. However, in this case we can obtain our result by use of Hilbert's double-integral theorem.†

$$(13.5) \qquad \begin{aligned} L_{1,t}[f(x)] &= \int_0^\infty \frac{u\phi(u)}{(t+u)^2} du, \\ \int_0^\infty |L_{1,t}[f(x)]|^p dt &\leqq r^p \int_0^\infty |\phi(u)|^p du, \end{aligned}$$

where

$$r = \frac{\Gamma\left(2 - \frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right)}{\Gamma(2)}.$$

Clearly (13.3) also holds, since

$$\lim_{x \to 0+} x f(x) = \lim_{t \to 0+} \int_0^t \phi(u) du.$$

† See G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge, 1934, p. 229, Theorem 319.

Hence the necessity of (13.2) is established.

Conversely, we see that (13.2) implies, by Corollary 10.32, that

$$f(a) = \lim_{k\to\infty} \int_0^\infty \frac{L_{k,t}[f(x)]}{t+a}\,dt + \frac{A}{a} \qquad (a > 0),$$

$$A = \lim_{x\to 0+} xf(x).$$

Furthermore, (13.2) implies† the existence of a subset k_i of all the integers k and a function $\phi(t)$ of class L^p in $(0, \infty)$ such that

$$\lim_{i\to\infty} \int_0^\infty \frac{L_{k_i,t}[f(x)]}{t+a}\,dt = \int_0^\infty \frac{\phi(t)}{t+a}\,dt.$$

Hence

$$f(x) = \frac{A}{x} + \int_0^\infty \frac{\phi(t)}{x+t}\,dt \qquad (x > 0). \tag{13.6}$$

But A is zero by virtue of (13.3), so that the theorem is established.

Corollary 13.11. *Conditions* (13.2) *are necessary and sufficient that* $f(x)$ *should have the representation* (13.6) *with* $\phi(t)$ *satisfying* (13.1).

Corollary 13.12. *If* $f(x)$ *has the representation* (4.1), (13.1), *then*

$$\lim_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)]|^p dt = \int_0^\infty |\phi(t)|^p dt.$$

For, Fatou's lemma gives

$$\int_0^\infty |\phi(t)|^p dt \leqq \liminf \int_0^\infty |L_{k,t}[f(x)]|^p dt.$$

This combined with (13.4) gives the result.

14. **Continuation,** $p=1$. That Theorem 13.1 can not hold for $p=1$ one sees from Theorem 11.1. For this case we prove

Theorem 14.1. *A necessary and sufficient condition that* $f(x)$ *should have the form* (4.1) *with* $\phi(t)$ *of class L in* $(0, \infty)$ *is that the functions* $L_{k,t}[f(x)]$, $(k=1, 2, \cdots)$, *should all be of class L and that*

$$\lim_{k,l\to\infty} \int_0^\infty |L_{k,t}[f(x)] - L_{l,t}[f(x)]|\,dt = 0, \tag{14.1}$$

$$\lim_{x\to 0+} xf(x) = 0. \tag{14.2}$$

† See S. Banach, *Opérations Linéaires*, p. 130. The proof there given is easily extended to the case of an infinite interval.

If $f(x)$ has the form (4.1) with $\phi(t)$ of class L, then

$$\int_0^\infty |L_{k,t}[f(x)]|\, dt \leqq d_k \int_0^\infty dt \int_0^\infty \frac{t^{k-1}u^k}{(t+u)^{2k}} |\phi(u)|\, du$$

$$= \int_0^\infty u^k |\phi(u)|\, du \int_0^\infty \frac{t^{k-1}}{(t+u)^{2k}}\, dt \leqq \int_0^\infty |\phi(u)|\, du \qquad (k = 1, 2, \cdots),$$

so that the first part of our condition is necessary. For the second part we have

$$|L_{k,t}[f(x)] - \phi(t)| \leqq d_k \int_0^\infty \frac{t^{k-1}u^k}{(t+u)^{2k}} |\phi(u) - \phi(t)|\, du$$

$$= d_k \int_0^\infty \frac{u^k}{(u+1)^{2k}} |\phi(tu) - \phi(t)|\, du.$$

Hence

$$\int_0^\infty |L_{k,t}[f(x)] - \phi(t)|\, dt \leqq d_k \int_0^\infty \frac{u^k}{(u+1)^{2k}} g(u)du,$$

where

$$g(u) = \int_0^\infty |\phi(tu) - \phi(t)|\, dt.$$

But $g(1) = 0$, $g(t)$ is continuous at $u = 1$, and a constant M exists such that

$$|g(u)| < Mu^{-1} + M \qquad (0 < u < \infty).$$

Under these conditions

$$\lim_{k\to\infty} d_k \int_0^\infty \frac{u^k}{(u+1)^{2k}} g(u)du = g(1) = 0$$

by Corollary 4.11. From this (14.1) is immediate.

Conversely, the assumption (14.1) implies the existence of a function $\phi(t)$ of class L such that

$$\lim_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)] - \phi(t)|\, dt = 0, \tag{14.3}$$

$$\lim_{k\to\infty} \int_0^\infty |L_{k,t}[f(x)]|\, dt = \int_0^\infty |\phi(t)|\, dt. \tag{14.4}$$

Equation (14.4) combined with (14.1) implies (11.1) for a suitable constant M. Hence by Theorem 11.1

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

where $\alpha(t)$ is of bounded variation in $(0, \infty)$. But

$$\left| \int_0^u L_{k,t}[f(x)]dt - \int_0^u \phi(t)dt \right| \leqq \int_0^\infty | L_{k,t}[f(x)] - \phi(t) | \, dt \qquad (u > 0).$$

Hence by (14.3)

$$\lim_{k\to\infty} \int_0^u L_{k,t}[f(x)]dt = \int_0^u \phi(t)dt.$$

But by Theorem 3.1

$$\lim_{k\to\infty} \int_0^u L_{k,t}[f(x)]dt = \alpha(u) - \alpha(0+) \qquad (u > 0).$$

By (14.2)

$$\lim_{x\to 0+} xf(x) = \alpha(0+) = 0,$$

so that

$$\alpha(u) = \int_0^u \phi(t)dt \qquad (u \geqq 0).$$

This completes the proof of the theorem.

COROLLARY 14.11. *If $f(x)$ has the form* (4.1) *with $\phi(t)$ of class L, then*

$$\lim_{k\to\infty} \int_0^\infty L_{k,t}[f(x)]dt = \int_0^\infty \phi(t)dt.$$

15. **$\phi(t)$ bounded.** To conjecture a condition for this case one would naturally allow p to become infinite in (13.2). This would lead to

$$| L_{k,t}[f(x)] | < N \qquad (k = 1, 2, \cdots). \tag{15.1}$$

But note that for $k=1$ we have (13.5) and that

$$\lim_{p\to\infty} r = \lim_{p\to\infty} \Gamma\left(2 - \frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right) = \infty.$$

In fact (15.1) is not necessary for the boundedness of $\phi(t)$. For, let $\phi(t)$ be equal to unity in (0, 1) and zero elsewhere. Then

$$L_{1,t}[f(x)] = \log\left(1 + \frac{1}{t}\right) - \frac{1}{t+1} \qquad (t > 0),$$

and this function becomes infinite as t approaches zero. We may overcome this difficulty by replacing (15.1) ($k=1$) by a condition on $f(x)$ of a slightly different type. The result is stated in

THEOREM 15.1. *A necessary and sufficient condition that $f(x)$ can be represented in the form* (4.1) *with $\phi(t)$ bounded is that*

$$|L_{k,t}[f(x)]| < M \qquad (t > 0;\ k = 2, 3, \cdots), \tag{15.2}$$

$$\lim_{x\to 0} xf(x) = 0, \tag{15.3}$$

$$\lim_{x\to\infty} f(x) = 0 \tag{15.4}$$

for a suitable constant M.

If $f(x)$ has the form (4.1), and if

$$|\phi(t)| < M \qquad (0 < t < \infty),$$

then

$$|L_{k,t}[f(x)]| \leqq d_k \int_0^\infty \frac{u^k t^{k-1}}{(u+t)^{2k}} |\phi(u)|\, du \quad (k = 2, 3, \cdots),$$

so that (15.2) is satisfied. Also

$$\lim_{x\to 0+} \int_0^\infty \frac{x\phi(t)}{x+t}\, dt = \lim_{u\to 0+} \int_0^u \phi(t) dt = 0,$$

$$\lim_{x\to\infty} \int_0^\infty \frac{\phi(t)}{x+t}\, dt = 0,$$

so that (15.3) and (15.4) also hold.

Conversely, (15.2) implies (10.2) at least for $k=2, 3, \cdots$. Also (15.3) and (15.4) imply (10.2) for $k=1$. Hence (10.3) and (10.4) hold. But these combined with (15.3) and (15.4) give

$$f^{(k)}(x) = o\left(\frac{1}{x^{k+1}}\right) \qquad (x \to 0;\ k = 0, 1, 2, \cdots),$$

$$f^{(k)}(x) = o\left(\frac{1}{x^k}\right) \qquad (x \to \infty;\ k = 0, 1, 2, \cdots).$$

Hence we obtain by successive integration by parts the identity

$$c_k(-1)^{k-1} \int_0^\infty \frac{\left[t^{2k-1} f^{(k-1)}(t)\right]^{(k)}}{(x+t)^2}\, dt = -\, d_k x^{k-2} \int_0^\infty \frac{t^{k+1}}{(x+t)^{2k}} f'(t) dt.$$

By (3.2)

$$-f'(x) = \lim_{k\to\infty} \int_0^\infty \frac{L_{k,t}[f(x)]}{(x+t)^2}\, dt.$$

Furthermore (15.2) implies the existence† of a subset k_i of the integers k and a bounded function $\phi(t)$ such that

$$-f'(x) = \lim_{i\to\infty} \int_0^\infty \frac{L_{k_i,t}[f(x)]}{(x+t)^2}\, dt = \int_0^\infty \frac{\phi(t)}{(x+t)^2}\, dt.$$

Now let $0<x<y$. Since the integral

$$\int_0^\infty \frac{\phi(t)}{(x+t)^2}\, dt$$

is uniformly convergent in any closed interval of the positive x-axis, we have

$$f(x) - f(y) = (y-x)\int_0^\infty \frac{\phi(t)}{(x+t)(y+t)}\, dt.$$

Hence, for any fixed positive x, (15.4) gives

$$\int_0^\infty \frac{\phi(t)}{(x+t)(y+t)}\, dt \sim \frac{f(x)}{y} \qquad (y\to\infty). \tag{15.5}$$

Moreover, since $\phi(t)$ is bounded, there exists a constant N such that

$$\frac{\phi(t)}{x+t} \geqq -\frac{N}{t} \qquad (0<t<\infty).$$

Hence we are in a position to apply a Tauberian theorem of Hardy and Littlewood‡ to the integral (15.5) considered as a function of y. The conclusion is that

$$f(x) = \int_0^\infty \frac{\phi(t)}{x+t}\, dt,$$

which is what we set out to prove.

Corollary 15.1. *If $f(x)$ has the form* (4.1) *with $\phi(t)$ bounded, then*§

$$\lim_{k\to\infty} \underset{0<t<\infty}{\text{l.u.b.}} \left| L_{k,t}[f(x)] \right| = \underset{0<t<\infty}{\text{true max}} \left| \phi(t) \right|.$$

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† See, for example, S. Banach, loc. cit., p. 130.

‡ G. H. Hardy and J. E. Littlewood, *Notes on the theory of series* (XI): *on Tauberian theorems*, Proceedings of the London Mathematical Society, vol. 30 (1930), p. 33.

§ For definition of true max see, for example, S. Banach, loc. cit., p. 227.

16. **A more general case.** We next investigate what functions $f(x)$ can be represented by a convergent integral of the form

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

with no restriction on $\alpha(t)$ except that it should be of bounded variation in every finite interval, and bounded in the infinite interval. To treat this case we need certain preliminary results which we now establish. We introduce a new operator $M_{k,t}[f(x)]$ by the

DEFINITION.

$$M_{k,t}[f(x)] = (-1)^{k-1}c_k[t^{2k-1}f^{(k-1)}(t)]^{(k-1)} \qquad (k = 2, 3, 4, \cdots),$$

$$M_{1,t}[f(x)] = tf(t).$$

Our first result is contained in

THEOREM 16.1. *If*

$$(16.1) \qquad f^{(n)}(t) = o\left(\frac{1}{t^{n+2}}\right) \qquad (t \to 0;\ n = 0, 1, 2, \cdots),$$

$$(16.2) \qquad f^{(n)}(t) = o\left(\frac{1}{t^n}\right) \qquad (t \to \infty;\ n = 0, 1, 2, \cdots),$$

then

$$f(x) = \lim_{k\to\infty} \int_0^\infty \frac{M_{k,t}[f(x)]}{(x+t)^2}\,dt \qquad (x > 0).$$

By an application of Theorem 9.1 or directly by integration by parts one shows that

$$\int_0^\infty \frac{M_{k,t}[f(x)]}{(x+t)^2}\,dt = d_k \int_0^\infty \frac{x^{k-1}t^k}{(x+t)^{2k}} f(t)dt.$$

Equation (3.2) now gives the result desired.

We shall next show that conditions (16.1), (16.2) follow automatically from the boundedness of $M_{k,t}[f(x)]$.

THEOREM 16.2. *If*

$$(16.3) \qquad M_{k,t}[f(x)] = O(1) \qquad (t \to \infty,\ t \to 0;\ k = 1, 2, \cdots),$$

then (16.1) *and* (16.2) *are true.*

For, one sees easily that (16.3) implies

$$f^{(k)}(t) = O\left(\frac{1}{t^{k+1}}\right) \qquad (t \to \infty,\ k = 0, 1, 2, \cdots),$$

of which (16.2) is a trivial consequence. Furthermore (16.3) implies

$$\text{(16.4)} \qquad \left[t^{2k-1}f^{(k-1)}(t)\right]^{(k-1)} = O(1) \qquad (t \to 0),$$

$$\text{(16.5)} \qquad \left[t^{2k-1}f^{(k-1)}(t)\right]^{(k-2)} = O(1) \qquad (t \to 0).$$

If we now assume (16.1) for $(n=0, 1, 2, \cdots, 2k-4)$ we see that it also holds for $n=2k-3$ by (16.5) and for $n=2k-2$ by (16.4). Since it obviously holds for $n=0$ by (16.4), $k=1$, it must hold in general.

By use of these results one may now prove

THEOREM 16.3. *A necessary and sufficient condition that*

$$\text{(16.6)} \qquad f(x) = \int_0^\infty \frac{\phi(t)}{(x+t)^2}\,dt,$$

where $\phi(t)$ is bounded is that there should exist a constant N for which

$$\text{(16.7)} \qquad \left| M_{k,t}[f(x)] \right| < N \qquad (k = 1, 2, 3, \cdots;\ 0 < x < \infty).$$

Note the contrast of this result with Theorem 15.1 by reason of the absence of any conditions of the type (15.3), (15.4). The proof follows by use of Theorems 16.1 and 16.2, and is omitted.

For the applications to follow it is desirable that the conditions of Theorem 16.3 involving the operator $M_{k,t}[f(x)]$ should be replaced by one involving $L_{k,t}[f(x)]$. We thus prove

THEOREM 16.4. *A necessary and sufficient condition that $f(x)$ should have the representation* (16.6) *with $\phi(t)$ bounded and satisfying*

$$\text{(16.8)} \qquad \int_0^t \phi(u)du \sim At \qquad (t \to 0)$$

for some constant A is that

$$\text{(16.9)} \qquad \left| \int_0^t L_{k,u}[f(x)]du \right| < N \qquad (k = 1, 2, 3, \cdots)$$

for some constant N.

If $f(x)$ has the representation described, then it follows from an abelian theorem, easily proved, that

$$\text{(16.10)} \qquad f^{(k)}(x) \sim \frac{(-1)^k k! A}{x^{k+1}} \qquad (x \to 0).$$

But (16.10) shows that

$$(16.11)\qquad \int_0^t L_{k,u}[f(x)]du = M_{k,t}[f(x)] - (-1)^{k-1}\left(\frac{k-1}{k}\right)A \qquad (k = 2, 3, \cdots),$$

$$(16.12)\qquad \int_0^t L_{1,u}[f(x)]du = M_{1,t}[f(x)] - A.$$

By Theorem 16.3 the right-hand sides of these equations have upper and lower bounds independent of k, so that the necessity of (16.9) is established.

Conversely, the existence of the integrals (16.9) implies by Theorem 10.2 the existence of a constant A such that

$$f^{(k)}(x) \sim \frac{(-1)^k A k!}{x^{k+1}} \qquad (x \to 0).$$

Hence (16.11) and (16.12) are again true and (16.9) implies (16.7). That is, (16.1) holds with $\phi(t)$ bounded. It remains only to establish (16.8). But this follows from (16.10), $k=0$, and the boundedness of $\phi(t)$ by a known Tauberian theorem.†

By use of these preliminary results we now prove

THEOREM 16.5. *A necessary and sufficient condition that*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t},$$

where $\alpha(t)$ is a normalized function of bounded variation in every finite interval and is bounded in the infinite interval, is that there should exist a constant M and a positive function $N(t)$ such that

$$(16.13)\qquad \left|\int_0^R L_{k,t}[f(x)]dt\right| \leqq M \qquad (R > 0;\ k = 1, 2, 3, \cdots),$$

$$(16.14)\qquad \int_0^R |L_{k,t}[f(x)]|\,dt \leqq N(R) \qquad (R > 0;\ k = 1, 2, 3, \cdots).$$

If $f(x)$ has the representation described, then

$$(16.15)\qquad f(x) = \int_0^\infty \frac{\alpha(t)}{(x+t)^2}dt,$$

$$\int_0^t \alpha(u)du \sim \alpha(0+)t \qquad (t \to 0),$$

† G. H. Hardy and J. E. Littlewood, *On Tauberian theorems*, Proceedings of the London Mathematical Society, vol. 30 (1930), p. 33, Theorem 5.

so that (16.3) follows from Theorem 16.4. Also

$$\lim_{k\to\infty} \int_0^R | L_{k,t}[f(x)] | \, dt = V(R) - V(0+)$$

from Theorem 6.3, so that the existence of $N(R)$ is insured.

Conversely, (16.13) implies that $f(x)$ has the form (16.15) with $\alpha(t)$ bounded and

$$\int_0^t \alpha(u) du \sim At \qquad (t \to 0).$$

Set

$$\alpha_k(t) = \int_0^t L_{k,u}[f(x)] du = M_{k,t}[f(x)] - (-1)^{k-1} \frac{k-1}{k} A.$$

But

$$M_{k,t}[f(x)] = d_k \int_0^\infty \frac{u^{k-1} t^k}{(t+u)^{2k}} \alpha(u) du,$$

and we showed in §3 that this integral approaches $\alpha(t)$ except perhaps in a set E of measure zero. But the variation of $\alpha_k(t)$ in $(0, R)$ is clearly not greater than $N(R)$ by (16.14). Hence $\alpha(t)$ is a normalized function of bounded variation in $(0, R)$ if suitably redefined. This redefinition has no effect on $f(x)$ since E is of measure zero. But

$$\int_0^\infty \frac{\alpha(t)}{(x+t)^2} dt = - \frac{\alpha(t)}{x+t} \Big|_0^\infty + \int_0^\infty \frac{d\alpha(t)}{x+t}.$$

The first term on the right-hand side is zero since $\alpha(t)$ is bounded and $\alpha(0) = 0$. Hence the theorem is completely established.

It might be supposed that we could remove the restriction of boundedness of $\alpha(t)$ by considering the function

$$F(x) = \frac{f(x) - f(\delta)}{\delta - x}$$

as was done in §12. For even if $\alpha(t)$ becomes infinite, $F(x)$ satisfies (16.13) and (16.14) when the integral

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} \tag{16.16}$$

converges. The converse is not true. If $F(x)$ satisfies (16.13) and (16.14) we have indeed

$$F(x) = \frac{f(x) - f(\delta)}{\delta - x} = \int_0^\infty \frac{\beta_\delta(t)}{(x+t)^2}\, dt,$$

where $\beta_\delta(t)$ is bounded. If $f(x)$ had the representation (16.16), then

$$\beta_\delta(t) = \int_0^t \frac{d\alpha(t)}{\delta + t}\cdot$$

But if $\alpha(t) = t \sin t$, then $\beta_\delta(t)$ is bounded. For

$$\beta_\delta(t) = \frac{t \sin t}{\delta + t} + \int_0^t \frac{u \sin u}{(\delta + u)^2}\, du,$$

and

$$\int_0^\infty \frac{u \sin u}{(\delta + u)^2}\, du$$

converges. But by Theorem 1.2 the integral (16.16) can not converge unless $\alpha(t) = o(t)$, which is not the case in the example considered.

17. **The Paley-Wiener inversion operator.** We conclude by showing the relation between the operator $L_{k,t}[f(x)]$ and the inversion operator given by Paley and Wiener.† They showed that if

$$f(x) = \int_0^\infty \frac{\phi(t)}{x + t}\, dt,$$

where $\phi(t)$ is a function of class L^2 in the interval $(0, \infty)$, then

$$\phi(t) = \underset{m\to\infty}{\text{l.i.m.}}\ \frac{1}{\pi t^{1/2}} \sum_{n=0}^{m} \frac{(-1)^n}{(2n)!} \left(\pi t \frac{d}{dt}\right)^{2n} (t^{1/2} f(t)).$$

We may abbreviate this precise result by the symbolic equation

$$\phi(t) = \frac{1}{\pi t^{1/2}} (\cos \pi \mathfrak{D})(t^{1/2} f(t)),$$

where

$$\mathfrak{D} = t \frac{d}{dt}\cdot$$

On the other hand

$$L_{k,t}[f(x)] = g_k \frac{1}{\pi t^{1/2}} \prod_{n=-k}^{k-2} \left(1 - \frac{2\mathfrak{D}}{2n+1}\right) (t^{1/2} f(t)), \tag{17.1}$$

where

† For reference see Introduction.

$$(17.2)\qquad g_k = \frac{\pi}{2}\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\dots\cdot\frac{2k-5}{2k-4}\cdot\frac{2k-3}{2k-4}\right)\frac{2k-3}{2k-2}\cdot\frac{2k-1}{2k}.$$

To prove this one has only to verify that the differential operators on opposite sides of equation (17.1) have the same system of fundamental solutions,

$$\frac{1}{x^k},\ \frac{1}{x^{k-1}},\ \dots,\ \frac{1}{x},\ 1,\ x,\ \dots,\ x^{k-3},\ x^{k-2},$$

and to compare the coefficients of $f^{(2k-1)}(t)$ in the expanded forms of both operators. This coefficient for the left-hand side of (17.1) is

$$c_k(-1)^{k-1}t^{2k-1},$$

while for the right-hand side it is

$$\frac{g_k}{\pi}\prod_{n=-k}^{k-2}\left(\frac{-2}{2n+1}\right)t^{2k-1}.$$

Equating these coefficients gives (17.2).

Since

$$\cos \pi z = \lim_{k\to\infty}\prod_{n=-k}^{k-2}\left(1-\frac{2z}{2n+1}\right),$$

and

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(\frac{2n-1}{2n}\cdot\frac{2n+1}{2n}\right),$$

it follows that

$$\lim_{k\to\infty} g_k = 1,$$

and that

$$\lim_{k\to\infty} L_{k,t}[f(x)] = \frac{1}{\pi t^{1/2}}(\cos \pi D)(t^{1/2}f(t)),$$

so that the two operators are symbolically equivalent.

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